The Eternal Art of Mathematics Continues to Inspire

(Examples of Tiling in the Alhambra from *Patronato de la Alhambra y Generalife*)

Religion, beauty, mathematics, and entertainment. Their intersection is not as empty as one might think. Those who have had the opportunity to play *Azul*, the 2018 winner of the *Speil Des Jahres* (Game of the Year)[1] have, inadvertently, experienced such an intersection. This beautifully crafted game credits its inspiration to the Portuguese ceramic tiles, called *azulejos*, which ultimately drew their inspiration from the Alhambra palace nestled in the Andalusian city of Granada. But what was it that initially inspired the architects of the Alhambra to create patterns that would resonate with artists and game designers for centuries. And what does such an inspiration have to do with mathematics?

Andalusia is a region in southern Spain that was occupied by the Moors, or the Muslim inhabitants of the western Mediterranean, from the 8th century through the 13th century. After its capital, Córdoba, was captured by Christians, the leader of the last Muslim dynasty on the Iberian Peninsula agreed to concede tribute and even territory to the Christians on the condition that they leave his hometown of Granada alone. Construction of the Alhambra began in 1238 and was completed later in the 1300s as the Muslim dynasty exhaled its last breath on the Iberian Peninsula. Its décor is distinctly different from the Christian basilicas of the time. While the basilicas were adorned with golden statues of saints, crucifixion displays, and murals of Biblical

[1] https://www.spiel-des-jahres.com/en/awardwinners. It turns out there is another *Speil Des Jahres* about the Alhambra simply called *Alhambra* that won the same prize 15 years ago.

tales, the Moors were strictly prohibited from depictions of Mohammed or Allah or any living creatures because it was seen as a form of idolatry. Without the expression of anthropomorphic representation, the Moors turned to represent the beauty of God in an abstract way.

The walls were adorned with intricate repeating patterns and tiling that held various visual symmetries. The fascinating result is that their numerous patterns and tessellations that are scaled across the walls of the palace contain rich mathematical properties, unique to the Alhambra, that have, in turn, been an inspiration to art. The ubiquitous tessellations of Dutch artist M.C. Escher were famously inspired by these wallpaper patterns.

But isn't mathematics about numbers with operations like addition and subtraction, and not artistic patterns? Rest assured, anywhere you see patterns or structure, there is mathematics lurking. In mathematics, there is a very precise description of what a symmetry means. Loosely, given an object, say an infinite wall with repeating designs, if you can rearrange the object in some way (think translate, rotate, and reflect in the case of our infinite wall) that makes it indistinguishable from its original state, then that rearrangement constitutes a symmetry. The branch of mathematics that studies symmetry is known as group theory and the set of all operations that leave the object unchanged constitutes a group. While operations in arithmetic are things like addition and subtraction, operations in group theory are rearrangements that don't alter the object. In our case, the three operations of translation, rotation, and reflection generate all rigid motions, that is, motions that preserve the distances between any two given points. In the plane, our infinite wall, if you have repeating patterns such as those depicted in the Alhambra or on the wallpaper in your living room, you can perform these rigid motions in certain ways such that, after performing these operations, the original state is left unaltered. With a few natural assumptions, it turns out that the mathematical structures that arise from these "wallpaper groups" can be classified into exactly 17 categories that encapsulates all possible rearrangements of the wall in terms of our rigid motions. There remains dispute as to exactly how many of these groups are present in the Alhambra, but the number ranges between 13 and 17, still a rich and unique accomplishment in architecture, especially given that it was produced well before group theory was invented.

One can play this game in higher dimensions as well. Due to the fact that they have numerous applications to crystal structures in three dimensions, these groups are often known as crystallographic groups and have significant applications in chemistry. Although we inhabit a three-dimensional world, however, our visual interests are peaked by the two-dimensional wallpaper groups. This is a result of the fact that we mentally construct our world based on a two-dimensional projection of our three-dimensional world. In three-dimensions, the number of categories jumps to 230. The number continues to balloon as the dimensions increase. Since our cognitive faculties only allow for visual representations of objects up to three dimensions, the beautiful symmetries involved in higher dimensions can only be "seen" through mathematical analysis. Given the rate of growth of the number of distinct crystallographic groups, the precise size for dimension six and up is still unknown. With recent advances in computational power and continuing mathematical research, maybe we can come to know more about these crystallographic symmetries in higher dimensions and thus, slightly enhance our view of the beauty of God.

References:

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