One Parameter Semigroups in Two Complex Variables

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Preliminary Definition

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A **one parameter semigroup** for a monoid $(S, *)$ is a map $\phi : [0, \infty) \to S$, such that

1. $\phi(0) = I$.
2. $\phi(s + t) = \phi(s) * \phi(t)$.

Intertwining Maps in the Disk

Under very general conditions, for an analytic map $\phi : \mathbb{D} \to \mathbb{D}$, there is a domain Ω, either the plane or half-plane, a mapping σ of $\mathbb{D}$ into Ω and a 'model' linear fractional automorphism Φ of Ω such that

$$\sigma \circ \phi = \Phi \circ \sigma$$

We have the following commutative diagram

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\ \ \phi\ \ } & \mathbb{D} \\ \big\downarrow \sigma & & \big\downarrow \sigma \\ \Omega & \xrightarrow{\ \ \Phi\ \ } & \Omega \end{array}$$

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$$\sigma \circ \phi = \Phi \circ \sigma \to \sigma \circ \phi_n = \Phi_n \circ \sigma.$$

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$$\sigma \circ \phi(z) = 1 - \left(\frac{1}{2}z + \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2}z = \frac{1}{2}(1 - z) = \Phi \circ \sigma(z).$$

Linear Fractional Maps in Higher Dimensions

Definition
We say ϕ is a linear fractional map in $\mathbb{C}^N$ if

$$\phi(z) = \frac{Az + B}{\langle z, C \rangle + D}$$

where A is an $N \times N$ matrix, B and C are column vectors in $\mathbb{C}^N$, $D \in \mathbb{C}$, and $\langle \cdot, \cdot \rangle$ is the standard inner product.

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Recall that Jordan Canonical Form says we can 'factor'

$$m_\phi = S \Lambda S^{-1} \to m_{\phi_n} = (m_\phi)^n = \left(S \Lambda S^{-1}\right)^n = S \Lambda^n S^{-1}.$$

where the columns of S are (generalized) eigenvectors of m_ϕ and Λ is in Jordan form.

Showing $\phi_{t \circ s} = \phi_{t+s}$

Diagonizable Case:

$$m_{\phi_t} = m_\phi^t = S\Lambda^t S^{-1} = S \begin{pmatrix} \lambda_1^t & 0 & 0 \\ 0 & \lambda_2^t & 0 \\ 0 & 0 & \lambda_3^t \end{pmatrix} S^{-1}.$$

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We then have for $s,\ t \geq 0$,

$$m_{\phi_t} m_{\phi_s} = S\Lambda^t S^{-1} S\Lambda^s S^{-1} = S\Lambda^{t+s} S^{-1} = m_{\phi_{t+s}}$$

The Case of a $3x3$ Jordan Block

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Exercise: $\Lambda^t \Lambda^s = \Lambda^{t+s}$.

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$$= \frac{\begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}}{(-1, 2)^T (z_1, z_2) + 3} = \frac{Az + B}{\langle z, C \rangle + D}.$$

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Example Continued

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$$m_{\phi_t} = \begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{8} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{t}{2} & \frac{t(t-1)}{8} \\ 0 & 1 & \frac{t}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -1 & 2 & 1 \\ -4 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2-t^2}{2} & t & \frac{t^2}{2} \\ -t & 1 & t \\ -\frac{t^2}{2} & t & \frac{t^2+2}{2} \end{pmatrix}$$

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$$\phi_t(z_1, z_2) = \left(\frac{(2-t^2)z_1 + 2tz_2 + t^2}{-t^2 z_1 + 2tz_2 + t^2 + 2}, \frac{-2tz_1 + 2z_2 + 2t}{-t^2 z_1 + 2tz_2 + t^2 + 2} \right).$$

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It is a straightforward calculation to see that $\phi_0 = I$ and $\phi_1 = \phi$.

Let's Get Crazy

Let $\phi(z) = (\phi_1(z_1, z_2), \phi_2(z_1, z_2))$ where

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$$\phi_1(z) = \frac{15z_1 + z_2 + 1 + 4\sqrt{2z_2(z_1+1)} + 4\sqrt{2(1-z_1^2)} + 2\sqrt{z_2(1-z_1)}}{-z_1 + z_2 + 17 + 4\sqrt{2z_2(z_1+1)} + 4\sqrt{2(1-z_1^2)} + 2\sqrt{z_2(1-z_1)}}$$

$$\phi_2(z) = \frac{16z_2 - z_1 + 1 + 8\sqrt{z_2(1-z_1)}}{-z_1 + z_2 + 17 + 4\sqrt{2z_2(z_1+1)} + 4\sqrt{2(1-z_1^2)} + 2\sqrt{z_2(1-z_1)}.}$$

Let's Get Crazy II

Define the following:

$$
\begin{aligned}
A :=& 1024z_1 + 64t^2 z_2 + t^2(t+7)^2(1-z_1) + 256t\sqrt{2z_2(z_1+1)} \\
&+ 32t(t+7)\sqrt{2(1-z_1^2)} + 16t^2(t+7)\sqrt{z_2(1-z_1)} \\
B :=& 1024 + 64t^2 z_2 + t^2(t+7)^2(1-z_1) + 256t\sqrt{2z_2(z_1+1)} \\
&+ 32t(t+7)\sqrt{2(1-z_1^2)} + 16t^2(t+7)\sqrt{z_2(1-z_1)} \\
C :=& 64t^2(1-z_1) + 1024z_2 + 512t\sqrt{z_2(1-z_1)} \\
D :=& 1024 + 64t^2 z_2 + t^2(t+7)^2(1-z_1) + 256t\sqrt{2z_2(z_1+1)} \\
&+ 32t(t+7)\sqrt{2(1-z_1^2)} + 16t^2(t+7)\sqrt{z_2(1-z_1)}.
\end{aligned}
$$

Too Crazy

A calculation shows $\phi_t(z) = (\phi_{1_t}(z), \phi_{2_t}(z))$ where $\phi_{1_t}(z)$ and $\phi_{2_t}(z)$ are given by

$$\phi_{1_t}(z) = \frac{A}{B}$$

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A calculation shows that this is a one-parameter semigroup for $\phi : \mathbb{B}_2 \to \mathbb{B}_2$.

Thank You and Happy π Day!

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Questions?

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