Spectra of Composition Operators on the Unit Ball in $\mathbb{C}^2$

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Preliminary Definitions

Definition
If ϕ is a self-map of the unit ball $\mathbb{B}_N$ then the composition operator C_ϕ is the linear operator defined by

$$C_\phi f = f \circ \phi$$

for a function f analytic in $\mathbb{B}_N$.

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Definition
Let X be a Hilbert space and T a bounded linear operator from X into X. The spectrum of T is defined to be

$$\sigma(T) := \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible}\}.$$

Statement of the Problem

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- Solve the eigenvalue equation $C_\phi f = \lambda f$.

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  - The vectors are complex-valued analytic functions.

- Determine the spectrum $\sigma(C_\phi)$.
  - The vectors reside in the Hilbert space of analytic functions known as the Drury-Arveson space $H_d^2(\mathbb{B}_N)$.

The Drury-Arveson Space

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We define the Drury-Arveson space H_d^2 to be the RKHS on $\mathbb{B}_N$ with kernel

$$k(z, w) = k_w(z) = \frac{1}{1 - \langle z, w \rangle}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product.

The Fixed-Point Behavior of ϕ is Important

The properties of the composition operator C_ϕ depend on its fixed point behavior.

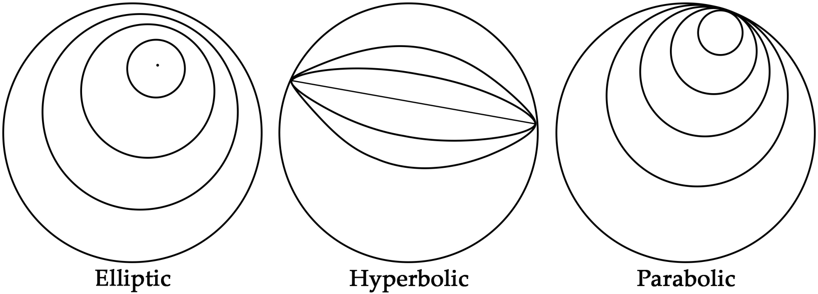

Figure 1: Automorphisms of the Disk

A Privileged Point

MacCluer (1982) demonstrated an analogue of the Denjoy Wolff Point for $\mathbb{B}_N$. In particular, suppose ϕ is a holomorphic, fixed point free self-map of $\mathbb{B}_N$. Then there exists a unique point ζ on the boundary such that the iterates of ϕ converge uniformly to ζ on compact subsets of $\mathbb{B}_N$.

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$$0 < d(\zeta) = \liminf_{z \to \zeta} \frac{1 - |\phi(z)|^2}{1 - |z|^2} = \alpha \leq 1.$$

When ζ is the Denjoy-Wolff point of ϕ, we call $d(\zeta)$ the dilation coefficient. By the Julia-Carathéodory Theorem, we have $d(\zeta) = \langle \phi'(\zeta)\zeta, \zeta \rangle$.

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Theorem
If ϕ is an analytic map of the ball $\mathbb{B}_N$ into itself that has no fixed points in the ball, then there is a unique fixed point ζ (the Denjoy-Wolff point) of ϕ on the boundary with $d(\zeta) \leq 1$. If $\phi(b) = b$ with $|b| = 1$, not the Denjoy-Wolff point, then $d(b) > 1$.

Classification by Dilation Coefficient

The dilation coefficient partitions our self maps of the ball into three classes.

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Definition
An analytic map ϕ from $\mathbb{B}_N$ into $\mathbb{B}_N$ is called

- elliptic if ϕ fixes an interior point of $\mathbb{B}_N$,
- hyperbolic if ϕ has no fixed point in $\mathbb{B}_N$ and dilation coefficient $\alpha < 1$
- parabolic if ϕ has no fixed point in $\mathbb{B}_N$ and dilation coefficient $\alpha = 1$.

A Little Historical Background About $C_\phi f = \lambda f$

- In 1884, Koenigs first solved this equation for an analytic map $\phi : \mathbb{D} \to \mathbb{D}$ such that $\phi(0) = 0$ and $\phi'(0) \neq 0$.

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- With some additional hypothesis, Enoch (2004) and Bridges (2012) extended this result to analytic maps of $\mathbb{B}_N$ to $\mathbb{B}_N$.
- Cowen (1981) includes consideration of the cases $\phi : \mathbb{D} \to \mathbb{D}$ when there is no interior fixed point in a uniform way by intertwining ϕ with linear fractional maps.

More on the Intertwining

Under very general conditions, for an analytic map $\phi : \mathbb{D} \to \mathbb{D}$, there is a domain Ω, either the plane or half-plane, a mapping σ of $\mathbb{D}$ into Ω and a 'model' linear fractional automorphism Φ of Ω such that

$$\sigma \circ \phi = \Phi \circ \sigma.$$

We have the following commutative diagram

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\quad \phi \quad} & \mathbb{D} \\ \Big\downarrow \sigma & & \Big\downarrow \sigma \\ \Omega & \xrightarrow{\quad \Phi \quad} & \Omega \end{array}$$

Definition
We say ϕ is a linear fractional map in $\mathbb{C}^N$ if

$$\phi(z) = \frac{Az + B}{\langle z, C\rangle + D}$$

where A is an $N \times N$ matrix, B and C are column vectors in $\mathbb{C}^N$, $D \in \mathbb{C}$, and $\langle \cdot, \cdot \rangle$ is the standard inner product.

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In our "hyperbolic" case, this extension admits two possible cases. One with domain $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \mid \Re(z_1) > 0\}$ (half-space) and the other with domain $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \mid \Re(z_1) > |z_2|^2\}$ (Siegel half-space).

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We are *presuming* that our analytic map ϕ has the above model. We will call these the half-space/dilation and Siegel half-space/dilation models.

Let's See a Simple Example

Example

Let $\phi(z) = \left(\frac{z_1+3}{4}, \frac{z_2}{2}\right)$. What is the Denjoy-Wolff point?

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Let $\phi(z) = \left(\frac{z_1+3}{4}, \frac{z_2}{2}\right)$. What is the Denjoy-Wolff point? To conform to our model we put $\sigma(z) = e_1 - z$ and $\Phi(z) = (\frac{1}{4}z_1, \frac{1}{2}z_2)$ which gives us

$$\sigma \circ \phi(z) = \left(1 - \frac{z_1 + 3}{4}, -\frac{z_2}{2}\right) = \left(\frac{1}{4}(1 - z_1), -\frac{1}{2}z_2\right) = \Phi \circ \sigma(z).$$

The Intertwining Strategy

Our strategy to solve the eigenvalue equation will be to start with the equation $F \circ \Phi = \lambda F$ and thus, letting $f = F \circ \sigma$, use the model results to determine the spectrum of the equation

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The "Hyperbolic" Solution (Condensed Version)

Theorem (Pilla)
Suppose ϕ *is an analytic map of the unit ball* $\mathbb{B}_2$ *into itself in the "hyperbolic" case such that* $\sigma \circ \phi = \Phi \circ \sigma$ *and* $\Phi(z) = (\alpha z_1, \beta z_2)$*.*

Theorem (Pilla)
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Then $f \circ \phi = \lambda f$ has a non-zero solution iff $\lambda = \alpha^k \beta^l$ for nonnegative integers k,l.

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Also f is a non-zero solution of $f \circ \phi = \lambda f$ iff $f(z) = c\sigma_1^k \sigma_2^l$ where c is a constant.

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Why aren't we done?

In the disk, we have

$$||C_\phi|| \leq \left(\frac{1 + |\phi(0)|}{1 - |\phi(0)|} \right)^{\frac{1}{2}}.$$

Boundedness Problems in Higher Dimensions

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In the ball $\mathbb{B}_N$ for $N > 1$, we don't even recover polynomials! (Wogen, 1987)

Positive Semi-Definite Functions

Definition
A function f is positive semi-definite if for any finite set $\{z_1, z_2, ..., z_n\}$ of distinct points and any complex numbers $c_1, c_2, ..., c_n$, we have

$$\sum_{i,j=1}^{n} \overline{c}_i c_j f(z_i, z_j) \geq 0.$$

The Schur-Agler Class

Definition
The Schur-Agler class S_n is the set of all holomorphic mappings $\phi : \mathbb{B}_n \to \mathbb{B}_n$ for which the Hermitian kernel

$$k^{\phi}(z, w) = \frac{1 - \langle \phi(z), \phi(w) \rangle}{1 - \langle z, w \rangle}$$

is positive semidefinite.

Spectral Radius

Theorem (Jury 2007)
If ϕ is in the Schur-Agler class, then $||C_\phi||$ is bounded on H^2_d and

$$\left(\frac{1}{1-|\phi(0)|^2}\right)^{\frac{1}{2}} \leq ||C_\phi|| \leq \left(\frac{1+|\phi(0)|}{1-|\phi(0)|}\right)^{\frac{1}{2}}.$$

Some Lemmas

Lemma (Circular Symmetry)
Suppose ϕ is an analytic map of $\mathbb{B}_2$ into itself in the "hyperbolic" case. Then for $\theta \in \mathbb{R}$ the operator C_ϕ acting on H^2_d is similar to the operator $e^{i\theta} C_\phi$.

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In particular, if $\lambda \in \sigma(C_\phi)$ then so is $\lambda e^{i\theta}$ for all real θ.

Lemma (Spectral Radius)
Let ϕ be in the Schur-Agler class and in our "hyperbolic" case with linear fractional model $\Phi(z) = (\alpha z, \beta z)$.

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In particular, if $\lambda \in \sigma(C_\phi)$ then so is $\lambda e^{i\theta}$ for all real θ.

Lemma (Spectral Radius)
Let ϕ be in the Schur-Agler class and in our "hyperbolic" case with linear fractional model $\Phi(z) = (\alpha z, \beta z)$.

Then the spectral radius of ϕ acting on H^2_d is $\alpha^{-\frac{1}{2}}$.

Lemma (Annulus of Eigenvalues)
Suppose ϕ is an analytic map from $\mathbb{B}_2$ into $\mathbb{B}_2$ in the Siegel half-space model with model map $\Phi(z) = (\alpha z_1, \beta z_2)$ and Denjoy-Wolff point $(1, 0)$. If

$$\alpha^{\frac{1}{2}} < |\lambda| < \alpha^{-\frac{1}{2}}$$

then λ is an eigenvalue of C_ϕ on H^2_d.

Some Lemmas

Lemma
Let ϕ be a map from the unit ball $\mathbb{B}_2$ into itself in the "hyperbolic" case, analytic in a neighborhood of the closed disk and not an inner function. Then $\sigma(C_\phi)$ intersects the circle of radius r for $0 < r < \alpha^{-\frac{1}{2}}$ (with conditions, currently).

The Main Result

Theorem (Pilla)
Assuming our hypotheses, for an analytic map $\phi : \mathbb{B}_2 \to \mathbb{B}_2$ with attracting boundary fixed point ζ, the spectrum of C_ϕ acting on $H^2_d(\mathbb{B}_2)$ is given by

$$\sigma(C_\phi) = \{\lambda \mid |\lambda| \leq \alpha^{-\frac{1}{2}}\}$$

where α is the radial limit of the complex directional derivative $D_\zeta \phi_\zeta$ at ζ.

Idea of Proof (In Words!)

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We have circular symmetry, so that $\sigma(C_\phi)$ includes the disk $\{\lambda \mid |\lambda| \leq \alpha^{-\frac{1}{2}}\}$.

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The spectral radius is $\alpha^{-\frac{1}{2}}$.

□

Return to Our Simple Example

Consider the function

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It is clear that $\phi : \mathbb{B}_2 \to \mathbb{B}_2$ has no fixed points in the ball and for $\zeta = (1, 0)$ we have $\phi(\zeta) = \zeta$ as the attracting fixed point with $D_\zeta \phi_\zeta(\zeta) = \frac{1}{4}$. Thus the spectrum of C_ϕ is given by

$$\sigma(C_\phi) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq \left(\frac{1}{4}\right)^{-\frac{1}{2}} = 2\}.$$

Further Work

- What can one say about the spectrum of the above considered composition operators on other Hilbert function spaces?

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- What can one say about the spectrum of the above considered composition operators on other Hilbert function spaces?
- What is the spectrum of composition operators with linear fractional models that are not in the "hyperbolic" case?
- What is the relationship between the hypotheses given and can you formulate a criteria to determine when an analytic self-map of the unit ball in $\mathbb{C}^2$ has a linear fractional model?

Thank you!

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Questions?

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References

R. Bridges. *A Solution to Schroeder's Equation in Several Variables.*
arXiv:1106.3370, (2011).

J. Cima and W. Wogen. *Unbounded Composition Operators on* $H^2(\mathbb{B}_2)$.
Proc. Amer. Math. Soc. 99 (1987) 477-483.

C. Cowen. *Iteration and the Solution of Functional Equations for Functions Analytic in the Unit Disk.*
Trans. Amer. Math. Soc. 265, no. 1, 69-95, (1981).

C. Cowen, D. Crosby, T. Horne, R. Ortiz Albino, A. Richman, Y. Yeow, B. Zerbe. *Geometric Properties of Linear Fractional Maps.*
Indiana University Mathematics Journal, Vol. 55m No. 2, pp.553-577 (2006).

C. Cowen and B. MacCluer. *Composition Operators on Spaces of Analytic Functions.*
CRC Press, Boca Raton, (1995).

References

R. Enoch. *Formal Power Series Solutions of Schroeder's Equation.*
Aequationes Mathematicae (2007).

M. Jury. *Reproducing Kernels, De Branges-Rovnyak Spaces, and Norms of Weighted Composition Operators.*
Proc. Amer. Math. Soc. Vol. 135, No. 11 (2007), 3669-3675.

G. Koenigs. *Recherches sur les intégrales de certaines équations fonctionelles.*
Annales scientifiques de l école Norm. Sup. (1884).

B. MacCluer. *Iterates of holomorphic self-maps of the unit ball in* $\mathbb{C}^N$ *.*
Michigan Mat. J. 30, no. 1 (1983), 97-106.