A Generalized Cross Ratio.

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Abstract. In one complex variable, the cross ratio is a well-known quantity associated with four given points in the complex plane that remains invariant under linear fractional maps. In particular, if one knows where three points in the complex plane are mapped under a linear fractional map, one can use this invariance to explicitly determine the map and to show that linear fractional maps are 3-transitive. In this paper, we define a generalized cross ratio and determine some of its basic properties. In particular, by defining linear fractional maps in several complex variables, we have a class of maps that obey similar transitivity properties as in one complex dimension, under some more restrictive conditions.

1. BACKGROUND. In 1872, Felix Klein famously introduced a point of view regarding what geometry should be about [**4**]. Known as the Erlangen program, Klein viewed geometry as a study of invariants under group transformations. An important and robust example of such an invariant quantity is given by the cross ratio.

Given four finite distinct points z_1, z_2, z_3, and z_4 in the complex plane, the cross ratio is defined as

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

If $z_i = \infty$ for some $i = 1, 2, 3, 4$ then we cancel the terms with z_i. For example, if $z_1 = \infty$, then

$$(z_1, z_2, z_3, z_4) = \frac{(z_2 - z_4)}{(z_2 - z_3)}.$$

With this definition, we may view the cross ratio as a function of z given by (z, z_1, z_2, z_3). Recall that a linear fractional map, also known as a Mobius transformation, is defined as

$$f(z) = \frac{az + b}{cz + d}$$

where the coefficients a, b, c, and d are complex numbers such that $ad - bc \neq 0$ (otherwise $f(z)$ is a constant function).

It is well known that the cross ratio is preserved under linear fractional maps. Thus if ϕ is a linear fractional map in one complex variable and $\phi(z_i) = w_i$ for $i = 1, 2, 3$ then

$$\frac{(w - w_2)(w_1 - w_3)}{(w - w_3)(w_1 - w_2)} = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}$$

where $w = \phi(z)$. In such a case one may solve for w to determine the map explicitly.

2. HOMOGENEOUS COORDINATES. Given a linear fractional map

$$\phi(z) = \frac{az+b}{cz+d}$$

we may define the associated matrix of ϕ as

$$m_\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Although linear fractional maps can be associated with matrices, they are clearly not linear transformations of $\mathbb{C}$ (take $f(z) = \frac{1}{z}$, for example). Instead, the associated matrix acts as a linear transformation on what are known as homogeneous coordinates in $\mathbb{C}^2$. Homogeneous coordinates are defined as follows. We associate the point $z = (z_1, z_2) \in \mathbb{C}^2$ where $z \neq \mathbf{0}$, the zero vector, with the point $\frac{z_1}{z_2} \in \mathbb{C}$. We will write $z \sim w$ if $z \in \mathbb{C}$ is the point associated with $w \in \mathbb{C}^2$. This space is equivalent to the Riemann sphere and is known as the complex projective line $\mathbf{CP}^1$. The point $(1, 0)$ is associated with the point at infinity on the Riemann sphere. Notice if $\phi(z) = w$ and the point $v \in \mathbf{CP}^1$ is associated with z, then $m_\phi v$ is associated with the point w and vice versa.

A linear transformation in $\mathbb{C}^2$ can be represented by a complex matrix as

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}.$$

Let $z \sim \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and $w \sim \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. Then we can associate the above linear transformation with the non-linear transformation

$$w = \frac{w_1}{w_2} = \frac{az_1 + bz_2}{cz_1 + dz_2} = \frac{a\left(\frac{z_1}{z_2}\right) + b}{c\left(\frac{z_1}{z_2}\right) + d} = \frac{az+b}{cz+d}.$$

Thus, the associated matrices are linear transformations acting on $\mathbf{CP}^1$! (see [4] for more details) A routine calculation shows that $m_{\phi_1 \circ \phi_2} = m_{\phi_1} m_{\phi_2}$ and $m_{\phi^{-1}} = (m_\phi)^{-1}$ as well.

We next define the cross ratio in homogeneous coordinates. First we define the following quantity. If $z = (z_1, z_2)$ and $w = (w_1, w_2)$ are points in $\mathbf{CP}^1$, then we define

$$[z, w] = \det \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix}.$$

We may then define the cross ratio as follows.

Definition. Let z_1, z_2, z_3, and z_4 be four distinct points in $\mathbf{CP}^1$, then we define the cross ratio to be

$$(z_1, z_2, z_3, z_4) = \frac{[z_1, z_3][z_2, z_4]}{[z_1, z_4][z_2, z_3]}.$$

This agrees with the above definition of the cross ratio. For finite points we can see this as follows. Given four finite distinct points z_1, z_2, z_3, and z_4 in $\mathbb{C}$, we write the associated points as $v_i = (z_i, 1)$ for $i = 1, 2, 3, 4$ and find

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{\det\begin{pmatrix} z_1 & z_3 \\ 1 & 1 \end{pmatrix} \det\begin{pmatrix} z_2 & z_4 \\ 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} z_1 & z_4 \\ 1 & 1 \end{pmatrix} \det\begin{pmatrix} z_2 & z_3 \\ 1 & 1 \end{pmatrix}}$$
$$= \frac{[v_1, v_3][v_2, v_4]}{[v_1, v_4][v_2, v_3]} = (v_1, v_2, v_3, v_4).$$

If one of our points is associated with the point at infinity, then without loss of generality, we let $z_1 = \infty$ and the point associated with z_1 is $v_1 = (1, 0)$ which gives us

$$(z_1, z_2, z_3, z_4) = \frac{(z_2 - z_4)}{(z_2 - z_3)} = \frac{\det\begin{pmatrix} z_2 & z_4 \\ 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} z_2 & z_3 \\ 1 & 1 \end{pmatrix}}$$
$$= \frac{\det\begin{pmatrix} 1 & z_3 \\ 0 & 1 \end{pmatrix} \det\begin{pmatrix} z_2 & z_4 \\ 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} 1 & z_4 \\ 0 & 1 \end{pmatrix} \det\begin{pmatrix} z_2 & z_3 \\ 1 & 1 \end{pmatrix}} = \frac{[v_1, v_3][v_2, v_4]}{[v_1, v_4][v_2, v_3]} = (v_1, v_2, v_3, v_4).$$

Thus, the cross ratio is not only invariant under ϕ but also m_ϕ. In addition to this, the definition of the cross ratio in terms of homogeneous coordinates is more unifying in the sense that the case of "infinity" is included in the definition. This resonates with Klein's Erlangen program and its focus on projective geometry (geometry done on projective spaces) as a more unifying geometry. In particular, we utilize this point of view to define a cross ratio in several complex variables.

3. LINEAR FRACTIONAL MAPS IN SEVERAL COMPLEX VARIABLES. To define the cross ratio in several complex variables, we must first say what it means to be a linear fractional map in $\mathbb{C}^N$ for $N > 1$. We saw that in one variable the linear fractional maps were not linear transformations on $\mathbb{C}$ but rather on the complex projective line $\mathbf{CP}^1$. In this paper, we take the perspective that we would like our linear fractional maps in $\mathbb{C}^N$ to be linear transformations on the complex projective space $\mathbf{CP}^N$.

Thus we associate the point $z = (z_1', z_2)$ where $z_1' \in \mathbb{C}^N$ and $z_2 \in \mathbb{C}$, $z \neq 0$, with the point $\frac{z_1'}{z_2} \in \mathbb{C}^N$. This associated space is known as the complex projective space

$\mathbf{CP}^N$. We now consider a linear transformation in $\mathbf{CP}^N$ which can be represented by a complex matrix as

$$\begin{pmatrix} A & B \\ C^* & D \end{pmatrix}$$

where A is an $N \times N$ matrix, B and C are column vectors in $\mathbb{C}^N$, and $D \in \mathbb{C}$. Denote the rows of A by a_i for $i = 1, ..., N$ and $B = \begin{pmatrix} b_1 & \cdots & b_N \end{pmatrix}^T$. For a point $\begin{pmatrix} z_1' \\ z_2 \end{pmatrix}$ in $\mathbf{CP}^N$, we have

$$\begin{pmatrix} w_1' \\ w_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix} \begin{pmatrix} z_1' \\ z_2 \end{pmatrix} = \begin{pmatrix} \langle a_1, \overline{z_1'} \rangle + b_1 z_2 \\ \vdots \\ \langle a_N, \overline{z_1'} \rangle + b_N z_2 \\ \langle z_1', C \rangle + D z_2 \end{pmatrix}.$$

Let $z \sim \begin{pmatrix} z_1' \\ z_2 \end{pmatrix}$ and $w \sim \begin{pmatrix} w_1' \\ w_2 \end{pmatrix}$. Then we can associate the above linear transformation in $\mathbf{CP}^N$ with the non-linear transformation in $\mathbb{C}^N$ given by

$$\begin{aligned} w = \frac{w_1'}{w_2} &= \left(\frac{\langle a_1, \overline{z_1'} \rangle + b_1 z_2}{\langle z_1', C \rangle + D z_2}, ..., \frac{\langle a_N, \overline{z_1'} \rangle + b_N z_2}{\langle z_1', C \rangle + D z_2} \right) \\ &= \left(\frac{\langle a_1, \overline{\frac{z_1'}{z_2}} \rangle + b_1}{\langle \frac{z_1'}{z_2}, C \rangle + D}, ..., \frac{\langle a_N, \overline{\frac{z_1'}{z_2}} \rangle + b_N}{\langle \frac{z_1'}{z_2}, C \rangle + D} \right) \\ &= \left(\frac{\langle a_1, \overline{z} \rangle + b_1}{\langle z, C \rangle + D}, ..., \frac{\langle a_N, \overline{z} \rangle + b_N}{\langle z, C \rangle + D} \right) \\ &= \frac{Az + B}{\langle z, C \rangle + D}. \end{aligned}$$

This is precisely the definition given by Cowen and MacCluer [**1**] and is the one we will adopt.

Definition. We say ϕ is a linear fractional map in $\mathbb{C}^N$ if

$$\phi(z) = \frac{Az + B}{\langle z, C \rangle + D}.$$

where A is an $N \times N$ matrix, B and C are column vectors in $\mathbb{C}^N$, $D \in \mathbb{C}$, and $\langle \cdot, \cdot \rangle$ is the standard inner product.

This class of maps has been studied in more generality by others [**3**], [**6**], [**7**], [**8**].

We define the associated matrix m_ϕ of the linear fractional map $\phi(z) = \frac{Az+B}{\langle z, C \rangle + D}$ to be given by

$$m_\phi = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix}$$

which, as we saw, is a linear transformation on $\mathbf{CP}^N$. If $\phi(z) = w$ and the point $v \in \mathbb{C}^N$ is associated with z, then $m_\phi v$ is associated with the point w and vice versa. A routine calculation also shows that $m_{\phi_1 \circ \phi_2} = m_{\phi_1} m_{\phi_2}$ and $m_{\phi^{-1}} = (m_\phi)^{-1}$. Thus, as we did in one variable, we can toggle back and forth between the complex space $\mathbb{C}^N$ and $\mathbf{CP}^N$ at our convenience.

4. THE CROSS RATIO IN SEVERAL COMPLEX VARIABLES. While generalized cross ratios have been defined previously in terms of the Schwarzian derivative [**2**], this paper takes a different perspective that is more tractable and attempts to be in the elementary spirit of the traditional cross ratio. For one complex variable, we saw that the cross ratio is invariant under the maps ϕ and m_ϕ. We would like the same to be true for the cross ratio in $\mathbb{C}^N$. The cross ratio can also be utilized to show that any three pairwise distinct points in $\mathbb{C}$ that are sent to another set of three pairwise distinct points uniquely determine a linear fractional map in $\mathbb{C}$. The proof of this uses the fact that one can utilize cross ratios to define a unique map that sends the pairwise distinct triple (z_1, z_2, z_3) to the standardized points $(1, 0, \infty)$. Generalizing to higher dimensions, however, does not come without cost. Even for $N = 2$, one recalls that projective transformations map lines to lines. Without further hypotheses to avoid problems of collinearity, the task of mapping a pairwise distinct quadruple (z_1, z_2, z_3, z_4) to four standardized points is hopeless. Treading carefully, we use these facts to motivate our definition of a cross ratio in $\mathbb{C}^N$ for $N > 1$. In particular we seek a minimal definition that will satisfy these requirements.

First we define the following quantity. If we let $z_i = (z_{1i}, z_{2i}, ..., z_{N+1,i})$ for $i = 1, ..., N+1$ be elements of $\mathbf{CP}^N$, then we define

$$[z_1, z_2, ..., z_{N+1}] = \det \begin{pmatrix} z_{11} & z_{12} & z_{13} & \cdots & z_{1,N+1} \\ z_{21} & z_{22} & z_{23} & \cdots & z_{2,N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{N+1,1} & z_{N+1,2} & z_{N+1,3} & \cdots & z_{N+1,N+1} \end{pmatrix}.$$

In order to motivate our general definition, we began by defining a cross ratio in $\mathbb{C}^2$ and proceed to generalize to $\mathbb{C}^N$.

Definition. Given 5 distinct points z_i for $i = 1, ..., 5$ in $\mathbf{CP}^2$, we define the cross ratio as

$$(z_1, z_2, z_3, z_4, z_5) = \frac{[z_1, z_3, z_5][z_2, z_4, z_5]}{[z_1, z_4, z_5][z_2, z_3, z_5]}.$$

One motivation for choosing this particular cross ratio is because it "reduces" to our usual cross ratio using the appropriate coordinates. That is, for the vector $z_i = (z_{1i}, z_{2i}, z_{3i})$ for $i = 1, .., 4$, if we choose local coordinates $z_{3i} = 1$, $x_i = \frac{z_{1i}}{z_{3i}}$, and $y_i = \frac{z_{2i}}{z_{3i}}$, we get vectors in $\mathbb{C}^2/\{0\}$ which we can interpret as $\mathbf{CP}^1$ and is sufficient to define the standard cross ratio. We still have one extra vector z_5. However, if we choose $z_5 = (0, 1, 0)$, a point on the line at infinity, in our local coordinates we have

$$(z_1, z_2, z_3, z_4, z_5) = \frac{[z_1, z_3, z_5][z_2, z_4, z_5]}{[z_1, z_4, z_5][z_2, z_3, z_5]}$$

$$= \frac{\det\begin{pmatrix} z_{11} & z_{13} & 0 \\ z_{21} & z_{23} & 1 \\ z_{31} & z_{33} & 0 \end{pmatrix} \det\begin{pmatrix} z_{12} & z_{14} & 0 \\ z_{22} & z_{24} & 1 \\ z_{32} & z_{34} & 0 \end{pmatrix}}{\det\begin{pmatrix} z_{11} & z_{14} & 0 \\ z_{21} & z_{24} & 1 \\ z_{31} & z_{34} & 0 \end{pmatrix} \det\begin{pmatrix} z_{12} & z_{13} & 0 \\ z_{22} & z_{23} & 1 \\ z_{32} & z_{33} & 0 \end{pmatrix}} = \frac{\det\begin{pmatrix} x_1 & x_3 & 0 \\ y_1 & y_3 & 1 \\ 1 & 1 & 0 \end{pmatrix} \det\begin{pmatrix} x_2 & x_4 & 0 \\ y_2 & y_4 & 1 \\ 1 & 1 & 0 \end{pmatrix}}{\det\begin{pmatrix} x_1 & x_4 & 0 \\ y_1 & y_4 & 1 \\ 1 & 1 & 0 \end{pmatrix} \det\begin{pmatrix} x_2 & x_3 & 0 \\ y_2 & y_3 & 1 \\ 1 & 1 & 0 \end{pmatrix}}$$

$$= \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)} = (x_1, x_2, x_3, x_4).$$

In $\mathbb{C}$, there are $4! = 24$ permutations of (z_1, z_2, z_3, z_4) of which 6 are distinct. In $\mathbb{C}^2$, we recognize that

$$(z_i, z_j, z_k, z_l, z_m) = (z_k, z_l, z_i, z_j, z_m) = (z_j, z_i, z_l, z_k, z_m) = (z_l, z_k, z_j, z_i, z_m)$$

and that no other permutation is equal to $(z_i, z_j, z_k, z_l, z_m)$. Thus, we have five ways to choose m and once m is chosen, we start with i (by the above equation we may, equivalently, start with any of i, j, k, l and have $3! = 6$ ways to choose the remaining three values which gives us $5 \times 6 = 30$ distinct permutations.

In addition to the cross ratio, it is constructive to also define a cross ratio pair.

Definition. We define the cross ratio pair in $\mathbf{CP}^2$ as

$$(z_1, z_2, z_3, z_4, z_5)_2 = \left(\frac{[z_1, z_3, z_5][z_2, z_4, z_5]}{[z_1, z_4, z_5][z_2, z_3, z_5]}, \frac{[z_1, z_3, z_4][z_2, z_4, z_5]}{[z_1, z_4, z_5][z_2, z_3, z_4]} \right)$$

where we see that this defines a linear fractional map when the point associated with z_1 is a variable in $\mathbb{C}^2$.

Motivated by the above results, we define a generalized cross ratio as follows:

Definition. Given $N + 3$ distinct points z_i for $i = 1, ...N + 3$ in $\mathbf{CP}^N$, we define the cross ratio as

$$(z_1, z_2, ..., z_{N+2}, z_{N+3}) = \frac{[z_1, z_3, z_5, ..., z_{N+3}][z_2, z_4, z_5, ..., z_{N+3}]}{[z_1, z_4, z_5, ..., z_{N+3}][z_2, z_3, z_5, ..., z_{N+3}]}$$

where each ellipsis represents the ordered sequence of omitted digits. To simplify the definition of the cross ratio N-tuple, we introduce new notation. We define $[z_i, z_j]^c_N$ by

$$[z_i, z_j]^c_N = [z_1, ..., z_{i-1}, z_{i+1}, ..., z_{j-1}, z_{j+1}, ...z_{N+3}].$$

We then define the cross ratio N-tuple as

$$(z_1, z_2, ..., z_{N+2}, z_{N+3})_N = \left(\left\{ \frac{[z_2, z_i]_N^c [z_1, z_3]_N^c}{[z_1, z_i]_N^c [z_2, z_3]_N^c} \right\}_{i=4,...,N+3} \right)$$

where the curly brackets denote a sequence over $i = 4, ..., N+3$ and we see that this defines a linear fractional map when the point associated with z_1 is a variable in $\mathbb{C}^N$.

Theorem 1. *Identifying local coordinates $x_{ij} = \frac{z_{ij}}{z_{N+1,j}}$ for $j = 1, ..., N$ and $z_{N+3} = (0, 0, ..., 0, 1, 0)^T$, the cross ratio in $\mathbf{CP}^N$ reduces to the cross ratio in $\mathbf{CP}^{N-1}$.*

Proof. We use proof by induction. We have seen that our result is true for $N = 2$. Now suppose $N = k+1$ where $k > 1$, making the appropriate identifications, we have

$$(z_1, z_2, ..., z_{(k+1)+3}) = \frac{[z_1, z_3, z_5, ..., z_{k+4}][z_2, z_4, z_5, ..., z_{k+4}]}{[z_1, z_4, z_5, ..., z_{k+4}][z_2, z_3, z_5, ..., z_{k+4}]}$$

$$= \frac{\det \begin{pmatrix} z_{1,1} & z_{1,3} & z_{1,5} & \cdots & z_{1,k+4} \\ z_{2,1} & z_{2,3} & z_{2,5} & \cdots & z_{2,k+4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{k+2,1} & z_{k+2,3} & z_{k+2,5} & \cdots & z_{k+2,k+4} \end{pmatrix} \det \begin{pmatrix} z_{1,2} & z_{1,4} & z_{1,5} & \cdots & z_{1,k+4} \\ z_{2,2} & z_{2,4} & z_{2,5} & \cdots & z_{2,k+4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{k+2,2} & z_{k+2,4} & z_{k+2,5} & \cdots & z_{k+2,k+4} \end{pmatrix}}{\det \begin{pmatrix} z_{1,1} & z_{1,4} & z_{1,5} & \cdots & z_{1,k+4} \\ z_{2,1} & z_{2,4} & z_{2,5} & \cdots & z_{2,k+4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{k+2,1} & z_{k+2,4} & z_{k+2,5} & \cdots & z_{k+2,k+4} \end{pmatrix} \det \begin{pmatrix} z_{1,2} & z_{1,3} & z_{1,5} & \cdots & z_{1,k+4} \\ z_{2,2} & z_{2,3} & z_{2,5} & \cdots & z_{2,k+4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{k+2,2} & z_{k+2,3} & z_{k+2,5} & \cdots & z_{k+2,k+4} \end{pmatrix}}$$

$$= \frac{\det \begin{pmatrix} x_{1,1} & x_{1,3} & x_{1,5} & \cdots & 0 \\ x_{2,1} & x_{2,3} & x_{2,5} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k+1,1} & x_{k+1,3} & x_{k+1,5} & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix} \det \begin{pmatrix} x_{1,2} & x_{1,4} & x_{1,5} & \cdots & 0 \\ x_{2,2} & x_{2,4} & x_{2,5} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k+1,2} & x_{k+1,4} & x_{k+1,5} & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}}{\det \begin{pmatrix} x_{1,1} & x_{1,4} & x_{1,5} & \cdots & 0 \\ x_{2,1} & x_{2,4} & x_{2,5} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k+1,1} & x_{k+1,4} & x_{k+1,5} & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix} \det \begin{pmatrix} x_{1,2} & x_{1,3} & x_{1,5} & \cdots & 0 \\ x_{2,2} & x_{2,3} & x_{2,5} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k+1,2} & x_{k+1,3} & x_{k+1,5} & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}}$$

$$= \frac{\det \begin{pmatrix} x_{1,1} & x_{1,3} & x_{1,5} & \cdots & x_{1,k+3} \\ x_{2,1} & x_{2,3} & x_{2,5} & \cdots & x_{2,k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k+1,1} & x_{k+1,3} & x_{k+1,5} & \cdots & x_{k+1,k+3} \end{pmatrix} \det \begin{pmatrix} x_{1,2} & x_{1,4} & x_{1,5} & \cdots & x_{1,k+3} \\ x_{2,2} & x_{2,4} & x_{2,5} & \cdots & x_{2,k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k+1,2} & x_{k+1,4} & x_{k+1,5} & \cdots & x_{k+1,k+3} \end{pmatrix}}{\det \begin{pmatrix} x_{1,1} & x_{1,4} & x_{1,5} & \cdots & x_{1,k+3} \\ x_{2,1} & x_{2,4} & x_{2,5} & \cdots & x_{2,k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k+1,1} & x_{k+1,4} & x_{k+1,5} & \cdots & x_{k+1,k+3} \end{pmatrix} \det \begin{pmatrix} x_{1,2} & x_{1,3} & x_{1,5} & \cdots & x_{1,k+3} \\ x_{2,2} & x_{2,3} & x_{2,5} & \cdots & x_{2,k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k+1,2} & x_{k+1,3} & x_{k+1,5} & \cdots & x_{k+1,k+3} \end{pmatrix}}$$

$$= \frac{[x_1, x_3, x_5, ..., x_{k+3}][x_2, x_4, x_5, ..., x_{k+3}]}{[x_1, x_4, x_5, ..., x_{k+3}][x_2, x_3, x_5, ..., x_{k+3}]} = (x_1, x_2, ..., x_{k+3}).$$

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We next show that this definition satisfies the invariant properties we would expect a cross ratio to have. Recall that for a nonzero parameter λ, the points $(\lambda z, \lambda w)$ and (z, w) in $\mathbf{CP}^2$ are associated with the same point in the complex plane. Which representative we choose, however, doesn't matter.

Theorem 2. *The cross ratio is independent of the representative chosen.*

Proof. Given $N+3$ nonzero parameters $\{\lambda_i\}_{i=1,\ldots,N+3}$, we have

$$\begin{aligned}
&(\lambda_1 z_1, \lambda_2 z_2, ..., \lambda_{N+2} z_{N+2}, \lambda_{N+3} z_{N+3}) \\
&= \frac{[\lambda_1 z_1, \lambda_3 z_3, \lambda_5 z_5, ..., \lambda_{N+3} z_{N+3}][\lambda_2 z_2, \lambda_4 z_4, \lambda_5 z_5, ..., \lambda_{N+3} z_{N+3}]}{[\lambda_1 z_1, \lambda_4 z_4, \lambda_5 z_5, ..., \lambda_{N+3} z_{N+3}][\lambda_2 z_2, \lambda_3 z_3, \lambda_5 z_5, ..., \lambda_{N+3} z_{N+3}]} \\
&= \frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5^2 ... \lambda_{N+3}^2 [z_1, z_3, z_5, ..., z_{N+3}][z_2, z_4, z_5, ..., z_{N+3}]}{\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5^2 ... \lambda_{N+3}^2 [z_1, z_4, z_5, ..., z_{N+3}][z_2, z_3, z_5, ..., z_{N+3}]} \\
&= \frac{[z_1, z_3, z_5, ..., z_{N+3}][z_2, z_4, z_5, ..., z_{N+3}]}{[z_1, z_4, z_5, ..., z_{N+3}][z_2, z_3, z_5, ..., z_{N+3}]} = (z_1, z_2, ..., z_{N+2}, z_{N+3})
\end{aligned}$$

■

By the same reasoning it is easy to see that the cross ratio pair is also independent of the representative chosen. For our next result we need the following lemma.

Lemma 1. *Given $N \times 1$ column vectors v_1, v_2,...,v_N, along with an invertible $N \times N$ matrix A, we have*

$$[Av_1, Av_2, ..., Av_N] = \det(A)[v_1, v_2, ..., v_N].$$

Proof. Denote the rows of A by a_i for $i = 1, 2, ..., N$ and $V = (v_1 \quad v_2 \quad \cdots \quad v_N)$. Then, since $\det(A)\det(V) = \det(AV)$, it suffices to show

$$A\,(v_1 \quad v_2 \quad \cdots \quad v_N) = (Av_1 \quad Av_2 \quad \cdots \quad Av_N)\,.$$

We then have

$$\begin{aligned}
A\,(v_1 \quad v_2 \quad \cdots \quad v_N) &= \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} (v_1 \quad v_2 \quad \cdots \quad v_N) = \begin{pmatrix} a_1 \cdot v_1 & a_1 \cdot v_2 & \cdots & a_1 \cdot v_N \\ a_2 \cdot v_1 & a_2 \cdot v_2 & \cdots & a_2 \cdot v_N \\ \vdots & \vdots & \ddots & \vdots \\ a_N \cdot v_1 & a_N \cdot v_2 & \cdots & a_N \cdot v_N \end{pmatrix} \\
&= \left(\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} v_1 \quad \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} v_2 \quad \cdots \quad \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} v_N \right) = (Av_1 \quad Av_2 \quad \cdots \quad Av_N)\,.
\end{aligned}$$

■

We next show that this cross ratio is invariant under associated matrices of linear fractional maps.

Theorem 3. *Let m_ϕ be the associated matrix to an invertible linear fractional map ϕ; that is, $\det m_\phi \neq 0$. Then the cross ratio is invariant under m_ϕ.*

Proof. Note by lemma 1 that $[m_\phi z_1, m_\phi z_2, ..., m_\phi z_k] = \det(m_\phi)[z_1, z_2, ..., z_k]$. Thus

$$\begin{aligned}
&(m_\phi z_1, m_\phi z_2, ..., m_\phi z_{N+3}) \\
&= \frac{[m_\phi z_1, m_\phi z_3, m_\phi z_5, ..., m_\phi z_{N+3}][m_\phi z_2, m_\phi z_4, m_\phi z_5, ..., m_\phi z_{N+3}]}{[m_\phi z_1, m_\phi z_4, m_\phi z_5, ..., m_\phi z_{N+3}][m_\phi z_2, m_\phi z_3, m_\phi z_5, ..., m_\phi z_{N+3}]} \\
&= \frac{\det(m_\phi)^2[z_1, z_3, z_5, ..., z_{N+3}][z_2, z_4, z_5, ..., z_{N+3}]}{\det(m_\phi)^2[z_1, z_4, z_5, ..., z_{N+3}][z_2, z_3, z_5, ..., z_{N+3}]} \\
&= \frac{[z_1, z_3, z_5, ..., z_{N+3}][z_2, z_4, z_5, ..., z_{N+3}]}{[z_1, z_4, z_5, ..., z_{N+3}][z_2, z_3, z_5, ..., z_{N+3}]} = (z_1, z_2, ..., z_{N+2}, z_{N+3})
\end{aligned}$$

where we can divide by $\det(m_\phi)$ since we presume m_ϕ is invertible. ∎

5. TRANSITIVITY OF LINEAR FRACTIONAL MAPS IN SEVERAL COMPLEX VARIABLES. It is well known that linear fractional maps acting on $\mathbb{C}$ are 3-transitive but not 4-transitive. That is, for six points z_1, z_2, z_3, w_1, w_2, and w_3 in $\mathbb{C}$ where the z_i's are pairwise distinct as are the w_i's, there is a unique linear fractional map ϕ such that $\phi(z_i) = w_i$ for $i = 1, 2, 3$. The usual way to achieve this result is to send the triple (z_1, z_2, z_3) to the standardized points $(1, 0, \infty)$ where ∞ represents the "north pole" on the Riemann sphere. In homogeneous coordinates, these standardized points are associated with $\begin{pmatrix}1\\1\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}$, and $\begin{pmatrix}1\\0\end{pmatrix}$, respectively. Thus, for the cross ratio as a function of z with associated point w given by $\phi(z) = \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}$, we may write the point associated with $\phi(z)$ as

$$m_\phi w = \begin{pmatrix}(z - z_2)(z_1 - z_3)\\(z - z_3)(z_1 - z_2)\end{pmatrix}.$$

It is clear that this sends our triple (z_1, z_2, z_3) to the points $(1, 0, \infty)$, respectively. Note that taking two distinct points z_1 and z_2 in $\mathbb{C}$ implies that the associated points are linearly independent in $\mathbb{C}^2$. However, it is not true that taking three distinct points in $\mathbb{C}^N$ implies the associated points will be linearly independent in $\mathbb{C}^{N+1}$. They could be collinear, for example. This fact will necessitate an additional assumption of linear independence that was not needed in $\mathbb{C}$.

We would like our cross ratio N-tuple to be a generalized formula for a multidimensional linear fractional map that maps $N + 2$ points to certain pre-specified points analogous to the triple $(1, 0, \infty)$. To achieve this, we consider an $N + 2$-tuple in which $N + 1$ of the points have linearly independent associated points which are sent to the standard basis vectors $\{e_i\}_{i=1}^{N+1}$ and the remaining associated point is sent to $\sum_{i=1}^{N+1} e_i$. To begin, we see how to do this in $\mathbb{C}^2$ and generalize.

Define the linear fractional map ϕ by

$$\phi(z) = \left(\frac{[z, z_3, z_5][z_2, z_4, z_5][z_2, z_3, z_4]}{[z, z_4, z_5][z_2, z_3, z_4][z_2, z_3, z_5]}, \frac{[z, z_3, z_4][z_2, z_3, z_5][z_2, z_4, z_5]}{[z, z_4, z_5][z_2, z_3, z_4][z_2, z_3, z_5]} \right).$$

Then the point w associated with $\phi(z)$ is given by

$$w = m_\phi z = \begin{pmatrix} [z, z_3, z_5][z_2, z_4, z_5][z_2, z_3, z_4] \\ [z, z_3, z_4][z_2, z_3, z_5][z_2, z_4, z_5] \\ [z, z_4, z_5][z_2, z_3, z_4][z_2, z_3, z_5] \end{pmatrix}.$$

We recall that $[z_i, z_j, z_k] = 0$ when any of i, j, k are equal or, more generally, there is linear dependence. In order to avoid these dependency problems, we presume $\{z_i\}_{i=3}^5$ forms a linearly independent set and that the triples $[\cdot, \cdot, \cdot]$ appearing in w with first entry z_2 are not zero, we make the assumption that $z_2 = \alpha z_3 + \beta z_4 + \gamma z_5$ where α, β, and γ are all non-zero. We will call this *the independence hypothesis*. Considering the three-dimensional space (α, β, γ), the independence hypothesis asks only that we omit a set of measure zero. We generalize this hypothesis to higher dimensions in the natural way.

With this in mind, it is clear to see that we have $m_\phi z_2 = (1, 1, 1)$, $m_\phi z_3 = (0, 0, 1)$, $m_\phi z_4 = (1, 0, 0)$, and $m_\phi z_5 = (0, 1, 0)$ as desired where $(1, 1, 1)$ is associated with $(1, 1)$, $(0, 0, 1)$ with $(0, 0)$, and vectors such as $(1, 0, 0)$, which we will denote by $e_{1,\infty}$, and $(0, 1, 0)$, which we will denote by $e_{2,\infty}$, correspond to the points in the hyperplane at infinity that are tangent to $(0, 1)$ and $(1, 0)$, respectively. In general, our goal will then be to map $N + 2$ points associated with $N + 2$ linearly independent vectors in $\mathbb{CP}^N$ to a set of $N + 2$ standardized points with associated vectors $\sum_{k=1}^{N+1} e_k$ and $\{e_k\}_{k=1}^{N+1}$ where e_i is the vector that is zero on every coordinate except the ith coordinate where it takes the value of 1.

Theorem 4. *Given $N + 2$ distinct elements w_2, w_3,...,w_{N+3} in $\mathbb{C}^N$ with respective associated points z_2, z_3,...,z_{N+3} in $\mathbb{CP}^N$ that satisfy the independence hypothesis, there exists a unique linear fractional map ϕ such that $m_\phi z_2 = (1, 1, ..., 1) = \sum_{k=1}^{N+1} e_k$, $m_\phi z_3 = e_{N+1}$, $m_\phi z_4 = e_1$, $m_\phi z_5 = e_2$, and $m_\phi z_i = e_{i-3}$ for $i > 5$.*

Proof. Define the linear fractional map ϕ by

$$\phi(z) = \left(\left\{ \frac{[z_2, z_i]^c_N [z, z_3]^c_N}{[z, z_i]^c_N [z_2, z_3]^c_N} \right\}_{i=4,\dots,N+3} \right).$$

We may then write the point associated with $\phi(z)$ as

$$w = \begin{pmatrix} \frac{[z_2,z_4]^c_N}{[z,z_4]^c_N} & \frac{[z_2,z_5]^c_N}{[z,z_5]^c_N} & \cdots & \frac{[z_2,z_3]^c_N}{[z,z_3]^c_N} \end{pmatrix}^T.$$

We recall that $[z_i, z_j]^c_N = [z, ..., z_{i-1}, z_{i+1}, ..., z_{j-1}, z_{j+1}, ...z_{N+3}] = 0$ when $z = z_k$ for some integer $1 < k < N + 3$, $k \neq i, j$. In particular, note that none of the denominators in the entries of w are zero. Thus we have $w(z_2) = \sum_{k=1}^{N+1} e_k$, $w(z_3) = e_{N+1}$, $w(z_4) = e_1$, $w(z_5) = e_2$, and $w(z_k) = e_{k-3}$.

We next demonstrate uniqueness. Suppose there are two linear fractional maps ϕ and ψ that send each w_i as above. Then the associated matrix $M = m_\phi m_\psi^{-1}$ fixes e_i and $v = \sum_{i=1}^N e_i$. Denote M by

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & a_{N3} & \dots & a_{NN} \end{pmatrix}.$$

Then $Me_i = e_i$ implies $a_{ij} = 0$ for $i \neq j$ and $Mv = v$ implies $a_{ii} = a_{jj}$ for all $1 \leq i, j \leq N$. We conclude that $M = a_{11}I$ where I is the identity which implies $m_\phi = a_{11}m_\psi$ and thus $\phi = \psi$. ■

To generalize in several variables, we use transitivity in a more restricted sense. In particular, we make the additional assumption that the initial data points satisfy the independence hypothesis. We next show that, in this sense of transitivity, linear fractional maps in $\mathbb{C}^N$ are $(N+2)$-transitive but not $(N+3)$-transitive. We began with some theorems.

Theorem 5. *Given $N+2$ pairwise distinct points $\{z_i\}_{i=1}^{N+2}$ in $\mathbb{C}^N$ with associated points $\{u_i\}_{i=1}^{N+2}$ and $N+2$ pairwise distinct points $\{w_i\}_{i=1}^{N+2}$ in $\mathbb{C}^N$ with associated points $\{v_i\}_{i=1}^{N+2}$ such that the sets $\{u_i\}_{i=1}^{N+2}$ and $\{v_i\}_{i=1}^{N+2}$ satisfy the independence hypothesis, there exists a unique linear fractional map $\chi(z)$ such that $\chi(z_i) = w_i$ for $i = 1, ..., N+2$.*

Proof. By Theorem 4, there exist maps ϕ and ψ such that $\phi(z_i) = e_i = \psi(w_i)$ for $i = 1, ..., N+1$ and $\phi(z_{N+2}) = \psi(w_{N+2}) = \sum_{k=1}^{N+1} e_k$. Define $m_\gamma = m_\psi^{-1} m_\phi$. This sends z_i to w_i for $i = 1, ..., N+2$. To show uniqueness, suppose m_σ maps z_i to w_i for $i = 1, ..., N+2$. Then m_ϕ and $m_\psi m_\sigma$ each send z_i to e_i for $i = 1, ..., N+2$. Thus by Theorem 4, we have $m_\phi = m_\psi m_\sigma$ which implies $m_\gamma = m_\psi^{-1} m_\phi = m_\sigma$ from which we conclude that $\gamma = \sigma$ as desired. ■

Theorem 6. *If ϕ fixes $N+2$ distinct points $\{z_i\}_1^{N+2}$ satisfying the independence hypothesis, then ϕ is the identity.*

Proof. Suppose that ϕ fixs $N+2$ distinct points $\{z_i\}_1^{N+2}$ satisfying the independence hypothesis, then, since the identity also fixes these points, by Theorem 4 we have that ϕ is equal to the identity. ■

We can thus conclude that our linear fractional maps are $(N+2)$-transitive but they are not $(N+3)$-transitive.

6. MORE TO EXPLORE. The author invites the reader to investigate this generalized definition further. For example, in one variable, we have a well-known criteria to determine when the cross ratio is real.

Theorem 7. *The cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle (where we let a line be a circle of infinite radius).*

Proof. Consider the cross ratio as a function of the first argument, $\phi(z) = (z, z_1, z_2, z_3)$. We saw that ϕ maps the triple (z_1, z_2, z_3) to $(1, 0, \infty)$, respectively. Since linear fractional maps send circles to circles, we conclude that ϕ sends our circle containing the triple (z_1, z_2, z_3) to the extended real line. Now consider the map $\phi^{-1}(\phi(z)) = z$. It is known that linear fractional maps are bijective and thus $\phi(z)$ is real if and only if $z = \phi^{-1}(\phi(z))$ lies on the circle containing our triple (z_1, z_2, z_3) which is the image of the extended real line under our map ϕ^{-1}. ■

In order for this theorem to extend to our generalized cross ratio N-tuple, it will take an upgrade of some geometric facts about these generalized maps. We appeal to results by Cowen and MacCluer (see [**1**], Section 3 for more details.). In particular, one finds more flexibility when working in higher dimensions. The appropriate generalization of circles in this situation is to consider *ellipsoids*, where an ellipsoid is a translate of the image of the unit ball under an invertible complex linear transformation. We recall Theorem 6 from [1].

Theorem 8. *If ϕ is a one-to-one linear fractional map defined on a ball $\overline{\mathbb{B}}$ in $\mathbb{C}^N$, then $\phi(\mathbb{B})$ is an ellipsoid.*

Proof. See [1], section 3. ■

This brings us to the following question.

Question. Is there an analogue to Theorem 7 for the generalized cross ratio if we relax the presumption of points lying on the boundary of a sphere and instead use the boundary of an ellipsoid?

In some form or another, this quantity and its special invariant properties have been examined and celebrated since antiquity. Now that we have a new perspective to talk about this quantity in higher dimensions, the author challenges the reader to explore what other similarities or differences the generalized version has.

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