A Generalized Cross Ratio

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Impossible Things

"*The cross-ratio does not generalize in a simple manner to higher dimensions*" - Wikipedia

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Definition
Given four finite distinct points z_1, z_2, z_3, and z_4 in the complex plane, the cross ratio is defined as

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

The Classical Cross Ratio

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If $z_i = \infty$ for some $i = 1, 2, 3, 4$ then we cancel the terms with z_i. For example, if $z_1 = \infty$, then

$$(z_1, z_2, z_3, z_4) = \frac{(z_2 - z_4)}{(z_2 - z_3)}.$$

Linear Fractional Maps

Definition
A linear fractional map, also known as a Mobius transformation, is defined as

$$\phi(z) = \frac{az + b}{cz + d}$$

where the coefficients a, b, c, and d are complex numbers such that $ad - bc \neq 0$.

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where the coefficients a, b, c, and d are complex numbers such that $ad - bc \neq 0$.

If ϕ is a linear fractional map in one complex variable and $\phi(z_i) = w_i$ for $i = 1, 2, 3$ then

$$\frac{(w - w_2)(w_1 - w_3)}{(w - w_3)(w_1 - w_2)} = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}.$$

Homogeneous Coordinates.

Given $w = (z_1, z_2) \in \mathbb{C}^{N+1}$ where $z_1 \in \mathbb{C}^N$ and $z_2 \in \mathbb{C}$ with $w \neq (0, 0)$, we identify $w \sim \frac{z_1}{z_2}$. Thus, homogeneous coordinates are scale invariant.

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Notice that in $\mathbb{C}$, the point at infinity is now nicely described by $(1, 0)$. In $\mathbb{C}^N$ for $N > 1$, this corresponds to 'a' point at infinity.

The Associated Matrix

For each LFM in $\mathbb{C}$, we define the associated matrix by

$$\phi(z) = \frac{az+b}{cz+d} \qquad \leftrightarrow \qquad m_\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

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Suppose $\widetilde{w} = m_\phi \widetilde{z}$ where $z \sim \widetilde{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and $w \sim \widetilde{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$.

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Then we have

$$w = \frac{w_1}{w_2} = \frac{az_1 + bz_2}{cz_1 + dz_2} = \frac{a\left(\frac{z_1}{z_2}\right) + b}{c\left(\frac{z_1}{z_2}\right) + d} = \frac{az+b}{cz+d}.$$

Thus, the associated matrices are linear transformations acting on the homogeneous coordinates in $\mathbb{C}$!

Linear Fractional Maps in $\mathbb{C}^N$.

Definition
A map ϕ is called a linear fractional map if

$$\phi(z) = \frac{Az + B}{\langle z, C \rangle + D}$$

where A is an $N \times N$ matrix, B and C are column vectors in $\mathbb{C}^N$, and $D \in \mathbb{C}$. (Cowen and MacCluer 2000)

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For each LFM in $\mathbb{C}^N$, we define the associated matrix by

$$\phi(z) = \frac{Az + B}{\langle z, C \rangle + D} \qquad \leftrightarrow \qquad m_\phi = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix}.$$

Relations Between a LFM and its Associated Matrix.

- From above we see that if $w = \phi(z)$ with $\widetilde{z} \sim z$ and $\widetilde{w} \sim w$, then $\widetilde{w} \sim m_\phi \widetilde{z}$ and vise-versa.

Relations Between a LFM and its Associated Matrix.

- From above we see that if $w = \phi(z)$ with $\widetilde{z} \sim z$ and $\widetilde{w} \sim w$, then $\widetilde{w} \sim m_\phi \widetilde{z}$ and vise-versa.
- A routine calculation shows that

$$m_{\phi_1 \circ \phi_2} = m_{\phi_1} m_{\phi_2} \quad \text{and} \quad m_{\phi^{-1}} = (m_\phi)^{-1}.$$

The Cross Ratio in Homogeneous Coordinates

Definition
If $z = (z_1, z_2)$ and $w = (w_1, w_2)$ are points in $\mathbf{CP}^1$, then we define

$$[z, w] = \det \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix}.$$

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$$[z, w] = \det \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix}.$$

Definition
Let w_1, w_2, w_3, and w_4 be four distinct points in $\mathbf{CP}^1$, then we define the cross ratio to be

$$(w_1, w_2, w_3, w_4) = \frac{[w_1, w_3][w_2, w_4]}{[w_1, w_4][w_2, w_3]}.$$

Demonstration of Equivalence

Given four finite distinct points z_1, z_2, z_3, and z_4 in $\mathbb{C}$, we write the associated points as $w_i = (z_i, 1)$ for $i = 1, 2, 3, 4$ and find

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{\det\begin{pmatrix} z_1 & z_3 \\ 1 & 1 \end{pmatrix} \det\begin{pmatrix} z_2 & z_4 \\ 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} z_1 & z_4 \\ 1 & 1 \end{pmatrix} \det\begin{pmatrix} z_2 & z_3 \\ 1 & 1 \end{pmatrix}}$$

$$= \frac{[w_1, w_3][w_2, w_4]}{[w_1, w_4][w_2, w_3]} = (w_1, w_2, w_3, w_4).$$

Demonstration of Equivalence

Given four finite distinct points z_1, z_2, z_3, and z_4 in $\mathbb{C}$, we write the associated points as $w_i = (z_i, 1)$ for $i = 1, 2, 3, 4$ and find

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$$= \frac{[w_1, w_3][w_2, w_4]}{[w_1, w_4][w_2, w_3]} = (w_1, w_2, w_3, w_4).$$

What happens if one of the points is infinite?

Let's Start Generalizing!

Definition
Let $z_i = (z_{1i}, z_{2i}, ..., z_{N+1,i})$ for $i = 1, ..., N+1$ be elements of $\mathbf{CP}^N$, then we define

$$[z_1, z_2, ..., z_{N+1}] = \det \begin{pmatrix} z_{11} & z_{12} & z_{13} & \cdots & z_{1,N+1} \\ z_{21} & z_{22} & z_{23} & \cdots & z_{2,N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{N+1,1} & z_{N+1,2} & z_{N+1,3} & \cdots & z_{N+1,N+1} \end{pmatrix}.$$

Definition
Given 5 distinct points z_i for $i = 1, ..., 5$ in $\mathbf{CP}^2$, we define the cross ratio as

$$(z_1, z_2, z_3, z_4, z_5) = \frac{[z_1, z_3, z_5][z_2, z_4, z_5]}{[z_1, z_4, z_5][z_2, z_3, z_5]}.$$

Why This Choice?

This choice"reduces" to our usual cross ratio using appropriate coordinates. For the vector $z_i = (z_{1i}, z_{2i}, z_{3i})$ for $i = 1, .., 4$, if we choose local coordinates $z_{3i} = 1$, $x_i = \frac{z_{1i}}{z_{3i}}$, and $y_i = \frac{z_{2i}}{z_{3i}}$, we get vectors in $\mathbb{C}^2/\{0\}$ which we can interpret as $\mathbf{CP}^1$ and is sufficient to define the standard cross ratio. We still have one extra vector z_5. Choose $z_5 = (0, 1, 0)$ to obtain

Why This Choice?

$$(z_1, z_2, z_3, z_4, z_5) = \frac{[z_1, z_3, z_5][z_2, z_4, z_5]}{[z_1, z_4, z_5][z_2, z_3, z_5]}$$

$$= \frac{\det \begin{pmatrix} z_{11} & z_{13} & 0 \\ z_{21} & z_{23} & 1 \\ z_{31} & z_{33} & 0 \end{pmatrix} \det \begin{pmatrix} z_{12} & z_{14} & 0 \\ z_{22} & z_{24} & 1 \\ z_{32} & z_{34} & 0 \end{pmatrix}}{\det \begin{pmatrix} z_{11} & z_{14} & 0 \\ z_{21} & z_{24} & 1 \\ z_{31} & z_{34} & 0 \end{pmatrix} \det \begin{pmatrix} z_{12} & z_{13} & 0 \\ z_{22} & z_{23} & 1 \\ z_{32} & z_{33} & 0 \end{pmatrix}}$$

Why This Choice?

$$(z_1, z_2, z_3, z_4, z_5) = \frac{[z_1, z_3, z_5][z_2, z_4, z_5]}{[z_1, z_4, z_5][z_2, z_3, z_5]}$$

$$= \frac{\det\begin{pmatrix} x_1 & x_3 & 0 \\ y_1 & y_3 & 1 \\ 1 & 1 & 0 \end{pmatrix} \det\begin{pmatrix} x_2 & x_4 & 0 \\ y_2 & y_4 & 1 \\ 1 & 1 & 0 \end{pmatrix}}{\det\begin{pmatrix} x_1 & x_4 & 0 \\ y_1 & y_4 & 1 \\ 1 & 1 & 0 \end{pmatrix} \det\begin{pmatrix} x_2 & x_3 & 0 \\ y_2 & y_3 & 1 \\ 1 & 1 & 0 \end{pmatrix}}$$

$$= \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)} = (x_1, x_2, x_3, x_4).$$

Definition

We define the cross ratio pair in $\mathbf{CP}^2$ as

$$(z_1, z_2, z_3, z_4, z_5)_2 = \left(\frac{[z_1, z_3, z_5][z_2, z_4, z_5]}{[z_1, z_4, z_5][z_2, z_3, z_5]}, \frac{[z_1, z_3, z_4][z_2, z_4, z_5]}{[z_1, z_4, z_5][z_2, z_3, z_4]} \right)$$

where we see that this defines a linear fractional map when the point associated with z_1 is a variable in $\mathbb{C}^2$.

Definition

Given $N+3$ distinct points z_i for $i = 1, ...N+3$ in $\mathbf{CP}^N$, we define the cross ratio as

$$(z_1, z_2, ..., z_{N+2}, z_{N+3}) = \frac{[z_1, z_3, z_5, ..., z_{N+3}][z_2, z_4, z_5, ..., z_{N+3}]}{[z_1, z_4, z_5, ..., z_{N+3}][z_2, z_3, z_5, ..., z_{N+3}]}$$

where each ellipsis represents the ordered sequence of omitted digits.

Definition
To simplify things, we introduce new notation. We define $[z_i, z_j]_N^c$ by

$$[z_i, z_j]_N^c = [z_1, ..., z_{i-1}, z_{i+1}, ..., z_{j-1}, z_{j+1}, ...z_{N+3}].$$

Generalizing to N Variables

Definition

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$$[z_i, z_j]_N^c = [z_1, ..., z_{i-1}, z_{i+1}, ..., z_{j-1}, z_{j+1}, ...z_{N+3}].$$

Definition

We define the cross ratio N-tuple as

$$(z_1, z_2, ..., z_{N+2}, z_{N+3})_N = \left(\left\{ \frac{[z_2, z_i]_N^c [z_1, z_3]_N^c}{[z_1, z_i]_N^c [z_2, z_3]_N^c} \right\}_{i=4,...,N+3} \right)$$

where the curly brackets denote a sequence over $i = 4, ..., N+3$.

Theorem
The cross ratio (and it's corresponding N-tuple) is independent of the representative chosen.

Some Theorems

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The cross ratio (and it's corresponding N-tuple) is independent of the representative chosen.

Theorem
Let m_ϕ be the associated matrix to an invertible linear fractional map ϕ; that is, $\det m_\phi \neq 0$. Then the cross ratio is invariant under m_ϕ.

Transitivity in $\mathbb{C}$

Recall LFMs in $\mathbb{C}$ are 3-transitive but not 4-transitive. That is, for six points z_1, z_2, z_3, w_1, w_2, and w_3 in $\mathbb{C}$ where the z_i's are pairwise distinct as are the w_i's, there is a unique linear fractional map ϕ such that $\phi(z_i) = w_i$ for $i = 1, 2, 3$.

The Strategy

Consider the map $\phi(z) = \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}$ where z_1, z_2, and z_3 are distinct. We may write the point associated with $\phi(z)$ as

$$m_\phi z = \begin{pmatrix} (z-z_2)(z_1-z_3) \\ (z-z_3)(z_1-z_2) \end{pmatrix}.$$

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$$m_\phi z = \begin{pmatrix} (z - z_2)(z_1 - z_3) \\ (z - z_3)(z_1 - z_2) \end{pmatrix}.$$

Where does this send the triple (z_1, z_2, z_3)?

Additional Assumptions

Taking two distinct points z_1 and z_2 in $\mathbb{C}$ implies that the associated points are linearly independent in $\mathbb{C}^2$.

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It is not true that taking three distinct points in $\mathbb{C}^N$ implies the associated points will be linearly independent in $\mathbb{C}^{N+1}$. This fact will necessitate an additional assumption of linear independence.

More Generalizing

Define the linear fractional map ϕ in $\mathbb{C}^2$ by

$$\phi(z) = \left(\frac{[z, z_3, z_5][z_2, z_4, z_5][z_2, z_3, z_4]}{[z, z_4, z_5][z_2, z_3, z_4][z_2, z_3, z_5]}, \frac{[z, z_3, z_4][z_2, z_3, z_5][z_2, z_4, z_5]}{[z, z_4, z_5][z_2, z_3, z_4][z_2, z_3, z_5]} \right).$$

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Then the point w associated with $\phi(z)$ is given by

$$w = m_\phi z = \begin{pmatrix} [z, z_3, z_5][z_2, z_4, z_5][z_2, z_3, z_4] \\ [z, z_3, z_4][z_2, z_3, z_5][z_2, z_4, z_5] \\ [z, z_4, z_5][z_2, z_3, z_4][z_2, z_3, z_5] \end{pmatrix}.$$

Where does this map send the quadruple z_2, z_3, z_4, z_5?

A Counterexample?

$$z_2 = (2, -1) \qquad \widetilde{z}_2 = (2, -1, 1)$$

$$z_3 = (0, 0) \qquad \widetilde{z}_3 = (0, 0, 1)$$

$$z_4 = (1, 0) \qquad \widetilde{z}_4 = (1, 0, 1)$$

$$z_5 = (0, 1) \qquad \widetilde{z}_5 = (0, 1, 1)$$

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The obstruction: $\widetilde{z}_2 = 0\widetilde{z}_3 + 2\widetilde{z}_4 - \widetilde{z}_5$ implies $\widetilde{z}_2$ is linearly dependent on $\widetilde{z}_3$ and $\widetilde{z}_4$, causing $[z_2, z_3, z_4]$ to equal zero.

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Independence Hypothesis: $\widetilde{z}_2 = \alpha\widetilde{z}_3 + \beta\widetilde{z}_4 + \gamma\widetilde{z}_5$ where α, β, γ are all non-zero.

Our Map

Using ϕ defined above, $m_\phi \widetilde{z}_2 = (1,1,1)$, $m_\phi \widetilde{z}_3 = (0,0,1)$, $m_\phi \widetilde{z}_4 = (1,0,0)$, and $m_\phi \widetilde{z}_5 = (0,1,0)$ as desired where $(1,1,1)$ is associated with $(1,1)$, $(0,0,1)$ with $(0,0)$, and vectors such as $(1,0,0)$, which we denote by $e_{1,\infty}$, and $(0,1,0)$, which we denote by $e_{2,\infty}$, correspond to the points in the hyperplane at infinity that are tangent to $(0,1)$ and $(1,0)$, respectively.

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Using similar methods as in one variable, it can be shown this map is unique.

Theorem

Given 4 pairwise distinct points $\{z_i\}_{i=1}^4$ in $\mathbb{C}^2$ with associated points $\{u_i\}_{i=1}^4$ and 4 pairwise distinct points $\{w_i\}_{i=1}^4$ in $\mathbb{C}^2$ with associated points $\{v_i\}_{i=1}^4$ such that the sets $\{u_i\}_{i=1}^4$ and $\{v_i\}_{i=1}^4$ satisfy the independence hypothesis, there exists a unique linear fractional map $\chi(z)$ such that $\chi(z_i) = w_i$ for $i = 1, ..., 4$.

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Theorem
If ϕ fixes 4 distinct points $\{z_i\}_1^4$ satisfying the independence hypothesis, then ϕ is the identity.

Some More Theorems

Theorem
Given 4 pairwise distinct points $\{z_i\}_{i=1}^4$ *in* $\mathbb{C}^2$ *with associated points* $\{u_i\}_{i=1}^4$ *and 4 pairwise distinct points* $\{w_i\}_{i=1}^4$ *in* $\mathbb{C}^2$ *with associated points* $\{v_i\}_{i=1}^4$ *such that the sets* $\{u_i\}_{i=1}^4$ *and* $\{v_i\}_{i=1}^4$ *satisfy the independence hypothesis, there exists a unique linear fractional map* $\chi(z)$ *such that* $\chi(z_i) = w_i$ *for* $i = 1, ..., 4$.

Theorem
If ϕ *fixes 4 distinct points* $\{z_i\}_1^4$ *satisfying the independence hypothesis, then* ϕ *is the identity.*

From this we can conclude that our LFMs in $\mathbb{C}^2$ are 4-transitive but not 5-transitive.

Generalizing to Any Dimensions

These results generalize in the natural way to $\mathbb{C}^N$ (Pilla 2020).

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Questions?

Thank you!

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