Random Walk on Three "Half-Cubes"

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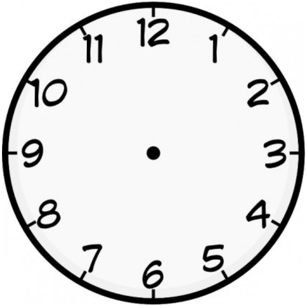
May 30, 2019

Random Walk on a Clock

Consider a Markov chain on the clock below that, at any point, jumps to one of its two adjacent numbers with equal probability.



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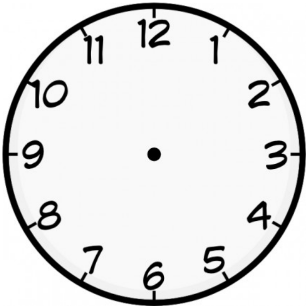

Q. Starting at noon, which vertex has the highest probability of being visited last?

Three Distinct Hemi-Cubes

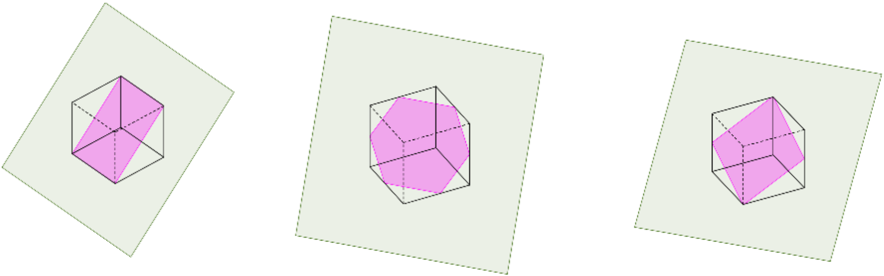

Figure: Three Topologically Distinct Hemi-Cubes: Rectangular, Hexagonal, and Rhombic, Respectively.

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Q2. Let L be the last vertex visited among all vertices (the last cookie eaten). What is $P(L = j | start = i)$?

Q1: Return Time on a Rectangular Hemi-Cube.

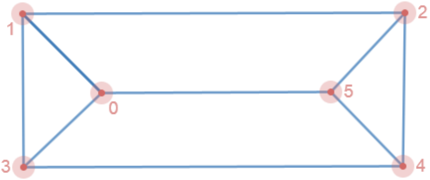

Figure: Planar Graph of Rectangular Hemi-Cube

By symmetry we may allow any point to be the origin. The RW starting at the origin goes to vertex 1, 3, or 5 with probability $\frac{1}{3}$. Thus we have

$$E(T_R) = 1 + \frac{1}{3}E({}_1T_0) + \frac{1}{3}E({}_3T_0) + \frac{1}{3}E({}_5T_0) = 1 + \frac{2}{3}E({}_1T_0) + \frac{1}{3}E({}_5T_0)$$

Q1: Return Time on a Rectangular Hemi-Cube

Likewise, we acquire the following formulas

- $E({}_1T_0) = 1 + \frac{1}{3}E({}_2T_0) + \frac{1}{3}E({}_3T_0)$
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Solving the systems of equations gives

- $E({}_1T_0) = E({}_3T_0) = \frac{24}{5}$
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- $E(T_R) = 6$

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One finds the following:

- $1 = 2P(L = 1) + 2P(L = 2) + P(L = 5)$
- $P(L = 1) = \frac{1}{3}(P(L = 2) + P(L_5 = 0, SL_5 = 1)) + \frac{1}{3}(P(L = 1) + P(L_3 = 0, SL_3 = 1))$
- $P(L = 5) = \frac{2}{3}(P(L = 2) + P(L_1 = 0, SL_1 = 5))$
[note, e.g., $(P(L_1 = 5) = P(L = 2)$]

To solve, we construct transition diagrams for $P(L_5 = 0, SL_5 = 1)$, $P(L_3 = 0, SL_3 = 1)$, and $P(L_1 = 0, SL_1 = 5)$.

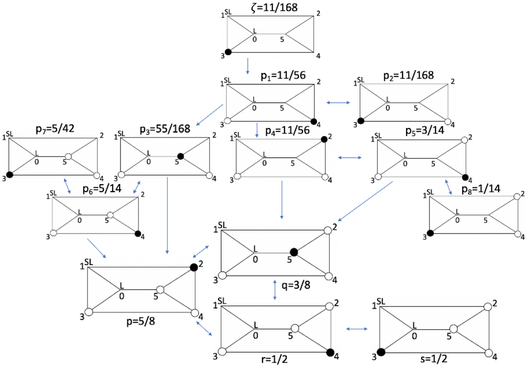

Figure: Transitiion Diagram for $P(L_3 = 0, SL_3 = 1)$

Q2: The Last Vertex Visited on a Rectangular Hemi-Cube

Solving the above equations using the transition diagrams gives

- $P(L = 1) = P(L = 3) = \frac{335}{1848} \approx 0.18127706$
- $P(L = 2) = P(L = 4) = \frac{29}{132} \approx 0.21969697$
- $P(L = 5) = \frac{61}{308} \approx 0.19805195$

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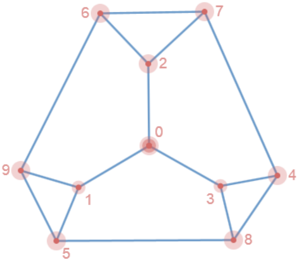
Matlab simulations (based on 10^6 iterations) gives 0.1816, 0.2195, 0.1981, respectively.

The Other Hemi-Cubes as Planar Graphs



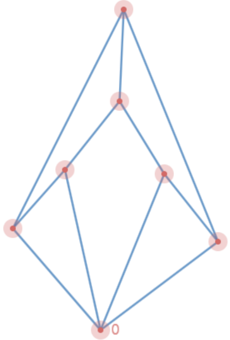


Figure: Hexagonal Hemi-Cube, Above
Rhombic Hemi-Cube, Right

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For the Hexigonal Hemi-Cube, if we start at the origin, we must solve the following system of equations:

- $1 = 3P(L = 1) + 6P(L = 4)$
- $P(L = 1) = \frac{2}{3}(P(L_2 = 1) + P(L_2 = 0, SL_2 = 1))$

Writing $P(L_2 = 1)$ in terms of other vertices gives a nook that leads to a (much) bigger opening. We find this generates five equations in five unknowns with twelve distinct terms of the form $P(L_i = j, SL_i = k)$, meaning we need twelve transition diagrams.

The Last Vertex Visited for the Hemi-Cube

Solving the appropriate equations using the twelve transition diagrams gives

- $P(L = 1) = \frac{36610785430753}{362875866925572} \approx 0.1008906592$
- $P(L = 4) = \frac{1940629641175501}{16692289878576312} \approx 0.1162590427$

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Matlab simulations (based on 10^8 iterations) gives 0.1009 and 0.1162, respectively.

Further Work

- In addition to determining the return time and last vertex visited, we are also completing a computation of the cover time and return to the origin after visiting all vertices.

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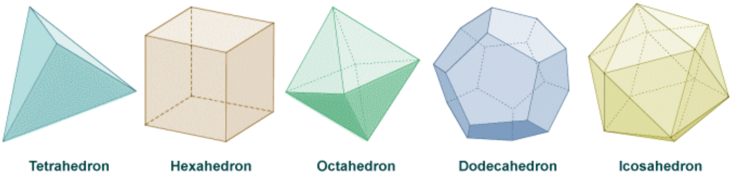
- In addition to determining the return time and last vertex visited, we are also completing a computation of the cover time and return to the origin after visiting all vertices.
- For the Platonic solids, results are known for the Tetrahedron, Cube, and Octahedron (Sarkar and Maiti, 2017). These questions remain unanswered for the Icosahedron and Dodecahedron.



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