Spectra of Composition Operators in $\mathbb{C}^2$

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Definition
If ϕ is a self-map of the unit ball $\mathbb{B}_N$ then the composition operator C_ϕ is the linear operator defined by

$$C_\phi f = f \circ \phi$$

for a function f analytic in $\mathbb{B}_N$.

Preliminary Definitions

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Definition
Let X be a Hilbert space and T a bounded linear operator from X into X. The spectrum of T is defined to be

$$\sigma(T) := \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible}\}.$$

Statement of the Problem

Goal:

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  - The vectors are complex-valued analytic functions.

- Determine the spectrum $\sigma(C_\phi)$.
  - The vectors reside in the Hilbert space of analytic functions known as the Drury-Arveson space $H^2_d(\mathbb{B}_N)$.

The Fixed-Point Behavior of ϕ is Important

The properties of the composition operator C_ϕ depend on its fixed point behavior.

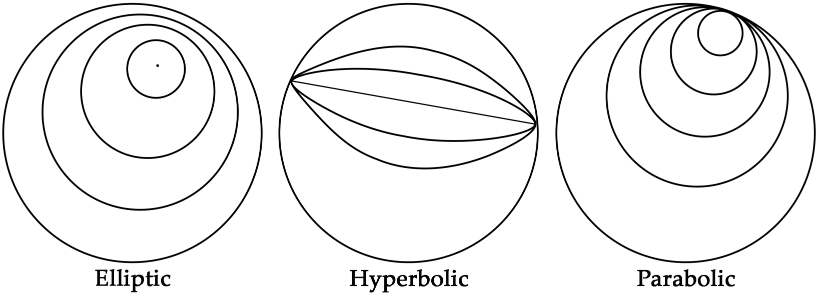

Figure 1: Automorphisms of the Disk

A Little Historical Background About $C_\phi f = \lambda f$

- In 1884, Koenigs first solved this equation for an analytic map $\phi : \mathbb{D} \to \mathbb{D}$ such that $\phi(0) = 0$ and $\phi'(0) \neq 0$.

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- With some additional hypothesis, Enoch (2004) and Bridges (2012) extended this result to analytic maps of $\mathbb{B}_N$ to $\mathbb{B}_N$.
- Cowen (1981) includes consideration of the cases $\phi : \mathbb{D} \to \mathbb{D}$ when there is no interior fixed point in a uniform way by intertwining ϕ with linear fractional maps.

More on the Intertwining

Under very general conditions, for an analytic map $\phi : \mathbb{D} \to \mathbb{D}$, there is a domain Ω, either the plane or half-plane, a mapping σ of $\mathbb{D}$ into Ω and a 'model' linear fractional automorphism Φ of Ω such that

$$\sigma \circ \phi = \Phi \circ \sigma.$$

We have the following commutative diagram

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\phi} & \mathbb{D} \\ \downarrow \sigma & & \downarrow \sigma \\ \Omega & \xrightarrow{\Phi} & \Omega \end{array}$$

Some Hypotheses for Higher Dimensions

In several variables, this is not adequate. However, one can show that the model theory extends in $\mathbb{B}_2$ for certain maps (Cowen, et al 2006).

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In our "hyperbolic" case, this extension admits two possible cases. One with domain $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \mid \Re(z_1) > 0\}$ (half-space) and the other with domain $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \mid \Re(z_1) > |z_2|^2\}$ (Siegel half-space).

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We are *presuming* that our analytic map ϕ has the above model. We will call these the half-space/dilation and Siegel half-space/dilation models.

The Main Result

Theorem (Pilla)
Assuming some natural hypotheses, for an analytic map $\phi : \mathbb{B}_2 \to \mathbb{B}_2$ in the Siegel half-space/dilation model with attracting boundary fixed point ζ, the spectrum of C_ϕ acting on $H^2_d(\mathbb{B}_2)$ is given by

$$\sigma(C_\phi) = \{\lambda \mid |\lambda| \leq \alpha^{-\frac{1}{2}}\}$$

where α is the radial limit of the complex directional derivative $D_\zeta \phi_\zeta$ at ζ.

A Simple Example

Consider the function

$$\phi(z) = \left(\frac{z_1 + 3}{4}, \frac{z_2}{2} \right).$$

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It is clear that $\phi : \mathbb{B}_2 \to \mathbb{B}_2$ has no fixed points in the ball and for $\zeta = (1, 0)$ we have $\phi(\zeta) = \zeta$ as the attracting fixed point with $D_\zeta \phi_\zeta(\zeta) = \frac{1}{4}$. Thus the spectrum of C_ϕ is given by

$$\sigma(C_\phi) = \{\lambda \in \mathbb{C} \mid |\lambda| < \left(\frac{1}{4}\right)^{-\frac{1}{2}} = 2\}.$$

Thank you!

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