A Classification of Linear Fractional Maps in Two Complex Variables

Michael R. Pilla

Advanced Topics Examination

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Linear Fractional Maps in $\mathbb{C}^N$.

Definition

A map ϕ is called a linear fractional map if

$$\phi(z) = \frac{Az + B}{\langle z, C \rangle + D}$$

where A is an $N \times N$ matrix, B and C are column vectors in $\mathbb{C}^N$, and $D \in \mathbb{C}$. (Cowen and MacCluer 2000)

Linear Algebra Link.

Definition

Given a linear fractional map $\phi(z) = \frac{Az+B}{\langle z,C\rangle+D}$ we define the associated matrix by

$$m_\phi = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix}.$$

Homogeneous Coordinates.

Given $v = (v_1, v_2) \in \mathbb{C}^{N+1}$ where $v_1 \in \mathbb{C}^N$ and $v_2 \in \mathbb{C}$ with $v \neq (0,0)$, we identify $v \sim \frac{v_1}{v_2}$. Thus, homogeneous coordinates are scale invariant.

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Notice that in $\mathbb{C}$, the point at infinity is now nicely described by $(1, 0)$. In $\mathbb{C}^N$ for $N > 1$, this corresponds to 'a' point at infinity.

The True Nature of the Associated Matrix.

A linear transformation in $\mathbb{C}^{N+1}$ can be represented by a complex $(N+1) \times (N+1)$ matrix as

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} Az_1 + Bz_2 \\ C^* z_1 + Dz_2 \end{pmatrix}.$$

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Let $z \sim \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and $w \sim \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. Then we can associate the above linear transformation to the non-linear transformation

$$w = \frac{w_1}{w_2} = \frac{Az_1 + Bz_2}{C^* z_1 + Dz_2} = \frac{A\left(\frac{z_1}{z_2}\right) + B}{C^*\left(\frac{z_1}{z_2}\right) + D} = \frac{Az + B}{C^* z + D}.$$

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Thus, the associated matrices are linear transformations acting on the homogeneous coordinates in $\mathbb{C}^{N+1}$!

Relations Between a Linear Fractional Map and its Associated Matrix.

- From above we see that if $\phi(z) = w$ with $v \sim z$ and $u \sim w$, then $m_\phi v \sim u$ and vise-versa.

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- From above we see that if $\phi(z) = w$ with $v \sim z$ and $u \sim w$, then $m_\phi v \sim u$ and vise-versa.
- A routine calculation shows that

$$m_{\phi_1 \circ \phi_2} = m_{\phi_1} m_{\phi_2} \quad \text{and} \quad m_{\phi^{-1}} = (m_\phi)^{-1}.$$

Eigenvalues and Eigenvectors.

Given a linear fractional map ϕ and its corresponding associated matrix m_ϕ, the eigenvalue equation looks like

$$m_\phi \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \iff$$

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and thus $z = (\frac{z_1}{z_3}, \frac{z_2}{z_3})$ is a fixed point of $\phi(z)$ if and only if $(z_1, z_2, z_3)^T$ is an eigenvector of m_ϕ.

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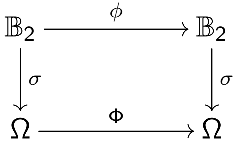
This conveniently dissolves the distinction between a given finite fixed point and a fixed point at infinity as the latter corresponds to the eigenvector of the form $(z_1, z_2, 0)^T$.

Classifying Maps.

We are looking for a small set of linear fractional maps Φ with domain Ω in $\mathbb{C}^2$ such that, given a linear fractional map ϕ of the ball into itself, we can intertwine it with a model map Φ and domain Ω and an open map σ from $\mathbb{B}_2$ into Ω such that

$$\sigma \circ \phi = \Phi \circ \sigma.$$



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$$\begin{array}{ccc} \mathbb{B}_2 & \xrightarrow{\ \phi\ } & \mathbb{B}_2 \\ \downarrow \sigma & & \downarrow \sigma \\ \Omega & \xrightarrow{\ \Phi\ } & \Omega \end{array}$$

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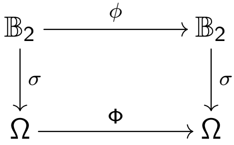

$$\sigma \circ \phi = \Phi \circ \sigma.$$

If Ω is the smallest set containing $\sigma(\mathbb{B}_2)$ for which $\Phi(\Omega) = \Omega$, then the model parameters (σ, Ω, Φ) will be unique up to holomorphic equivalence.

The Denjoy-Wolff Point: A Privileged Fixed Point.

Theorem

If ϕ is an analytic map of the ball into itself with no fixed points in B_N, then there is a fixed point ζ of norm 1 so that the iterates ϕ_n of ϕ converge to ζ uniformly on compact subsets of B_N. (See, for example, Cowen and MacCluer 1995)

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- We will call ζ the Denjoy-Wolff point.

The Case for Maps with an Interior Fixed Point.

A routine calculation shows that conjugation by an automorphism does not change our classification data. WLOG we may assume our Denjoy-Wolff point is at 0 or e_1. That is, if our Denjoy-Wolff point is in the ball, we may conjugate it to the origin. If our Denjoy-Wolff point is on the boundary, we may rotate the ball to relocate our point to the "east pole" at e_1.

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Theorem

If a linear fractional map ϕ has a Denjoy-Wolff point at 0, then there exists a linear fractional map σ such that $\sigma \circ \phi = \phi'(0)\sigma$. Thus $\Phi(z) = \phi'(0)z$ and $\Omega = \mathbb{C}^2$. (Cowen and MacCluer 2003)

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- With some extra hypotheses, Ruth Enoch (2004) and Bobby Bridges (2012) were able to extend this result to analytic maps of B_N into B_N.

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where the columns of S are eigenvectors of m_ϕ and Λ is in Jordan form.

- Note that if ψ is an automorphism of $\mathbb{B}_2$ then

$$m_\psi m_\phi m_\psi^{-1} = (m_\psi S)\Lambda(m_\psi S)^{-1}$$

and thus we have the same Jordan form.

Cast in the Light of Linear Algebra Part II.

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- For the case of three distinct fixed points, this will correspond to a diagonal matrix Λ.
- Thus we will relocate our fixed points to more convenient places by automorphisms, giving a simple form to S!
- One can show that m_ϕ has exactly one generalized eigenvector corresponding to a point which is not e_1-tangential. Note that the vectors in $\mathbb{C}^{N+1}$ whose first and last coordinates are zero correspond to the points in the hyperplane at infinity that are e_1-tangential.

The Plan.

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  This corresponds to the associated vector $(1, \beta, 0)^T$.
- If we have fixed point multiplicity at e_1 producing generalized eigenvectors in S, we just do the same automorphisms anyway.

The Heisenberg Translation.

- Our first automorphism is given by $\psi_b = \Psi^{-1} \circ h_b \circ \Psi$ where

$$\Psi(z) = \frac{z + e_1}{-z_1 + 1} \quad \text{and} \quad h_b(z) = Az + b. \tag{1}$$

with

$$A = \begin{pmatrix} 1 & 2\overline{b_2} \\ 0 & 1 \end{pmatrix}$$

where Ψ is a biholomorphic map from the ball onto the Siegel half-space $S = \{z \mid \mathrm{Re}(z_1) > |z_2|^2\}$ and $b = (b_1, b_2)$ is on the boundary of S.

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- Ψ maps e_1 to an infinite point in the direction of e_1 and $m_{h_b} e_{1,\infty} = e_{1,\infty}$. Likewise, $m_{h_b}(v, 0)^T = (Av, 0)^T$ tells us that finite points are mapped to finite points and the hyperplane at infinity is fixed as a set.

Mapping the Fixed Points.

- If the associated vector to the e_1-tangential fixed point is given by $(1, \gamma, 1)^T$ and we choose $\overline{b_2} = -\frac{1}{\gamma}$ for $\gamma \neq 0$, a simple calculation shows that our e_1-tangential fixed point is sent to the e_1 hyperplane at infinity. Then we replace S with $m_{\psi_b} S$.

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- Next we consider the automorphism

$$\tau_k = \begin{pmatrix} 1+|k|^2 & 0 & |k|^2-1 \\ 0 & 2k & 0 \\ |k|^2-1 & 0 & 1+|k|^2 \end{pmatrix}$$

which fixes e_1 and e_1-tangential points at infinity.

Mapping the Fixed Points II.

- We can choose k so that our remaining fixed point is sent to infinity along a non-e_1-tangential approach to give us $(1, \beta, 0)^T$ as our eigenvector provided our eigenvector is not associated with $-e_1$.

Mapping the Fixed Points II.

- We can choose k so that our remaining fixed point is sent to infinity along a non-e_1-tangential approach to give us $(1, \beta, 0)^T$ as our eigenvector provided our eigenvector is not associated with $-e_1$.
- If our third fixed point is $-e_1$, then we use the automorphism of $\mathbb{C}^N$ given by

$$m_\eta = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

where we see $\eta(\{z \mid -1 < \mathrm{Re}(z_1) < 1\}) = \{z \mid \mathrm{Re}(z_1) < 1\})$.

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  where we see $\eta(\{z \mid -1 < \mathrm{Re}(z_1) < 1\}) = \{z \mid \mathrm{Re}(z_1) < 1\}$.
- Thus, when choosing the appropriate automorphisms, we can obtain a new S in the form

$$S = \begin{pmatrix} 1 & 0 & 1 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A Theorem for Three Distinct Fixed Points.

Theorem

If ϕ is a linear fractional map which has a Denjoy-Wolff point on the boundary of $\mathbb{B}_2$ and which has three distinct fixed points, then the domain Ω for the model is a half-space or a Siegel half-space.

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Before we demonstrate this theorem, we note some other useful theorems and definitions.

Krein Inner Product.

Definition

We will call an inner product on $\mathbb{C}^{N+1}$ a Krein inner product if for u, $v \in \mathbb{C}^{N+1}$,

$$[u, v] = \langle Ju, v \rangle$$

where $J = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}$ and $\langle \cdot, \cdot \rangle$ is the usual Euclidean inner product on $\mathbb{C}^{N+1}$.

In this sense, we see that v represents a point of the unit sphere iff $|v_1| = |v_2|$ iff $[v, v] = |v_1|^2 - |v_2|^2 = 0$ and represents a point of the unit ball iff $[v, v] < 0$.

Useful Theorem 1 (UT1).

Theorem

A nonconstant linear fractional map ϕ maps B_2 into itself iff its associated matrix m_ϕ is a Krein contraction, i.e. there is a t with

$$[tm_\phi, tm_\phi v] \leq [v, v] < 0$$

for $v \sim z \in B_2$. (Cowen and MacCluer 2000)

Useful Theorem 2 (UT2).

Theorem

For $d = (1, 0, 1)^T$, the value of t above is given by

$$t = \sqrt{\frac{Re([d, x])}{Re([d, m_\phi x])}}$$

where $x \in \mathbb{C}^{N+1}$ is any vector such that $[d, x] \neq 0$. Additionally, $\lambda_1 = \frac{Re([d, m_\phi x])}{Re([d, x])}$ and thus $0 < \lambda_1 < 1$. (A. Richman 2002)

Proof of Theorem Part I.

Proof.

We presume e_1 is the Denjoy-Wolff point and m_ϕ is diagonalizable. We use our automorphisms ψ_b and τ_b to obtain

$$m_\phi = \begin{pmatrix} 1 & 0 & 1 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

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According to UT1 we have

$$\begin{aligned} [tm_\phi v, tm_\phi v] \leq [v, v] < 0 &\leftrightarrow \langle Jtm_\phi v, tm_\phi v\rangle \leq \langle Jv, v\rangle \\ &\leftrightarrow 0 \leq \langle (J - t^2 m_\phi^* J m_\phi) v, v\rangle \\ &\leftrightarrow J - t^2 m_\phi^* J m_\phi \text{is positive semi-definite.} \end{aligned}$$

□

Proof of Theorem Part II.

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By direct computation of $J - t^2 m_\phi^* J m_\phi$ in MATLAB and using Sylvester's Criterion, we find a necessary condition that $\lambda_1 \geq |\lambda_2|^2$.

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A direct computation of m_ϕ^n also yields
$\phi_n(z_1, z_2) = (\lambda_1^n(z_1 - 1) + 1, \beta(\lambda_1^n - \lambda_2^n)(z_1 - 1) + \lambda_2^n z_2)$.

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- If $\lambda_1 > |\lambda_2|^2$, this is in B_2 for sufficiently large n iff $\mathrm{Re}(z_1) < 1$. Thus gives a half-space/dilation model.
- If $\lambda_1 = |\lambda_2|^2$, this is in B_2 for sufficiently large n iff $\beta = 0$ and $\mathrm{Re}(z_1) < 2 - |z_2|^2$. This gives a Siegel half-space/dilation model.

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We can choose $\Phi(z) = Az$ with $|A| < 1$ and $\sigma(z) = e_1 - z$.

□

Cases of Multiplicity.

If our Denjoy-Wolff point at e_1 has multiplicity greater than 1, it is easier to work on the Siegel half-space. Recall the map Ψ does this. We are looking for automorphisms of the Siegel half-space that fix $e_{1,\infty}$. We find that these include Heisenberg translations, translations of the form $z + e_1$, and contractions of the second variable (z_1, cz_2) where $|c| \leq 1$.

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If our Denjoy-Wolff point at e_1 has multiplicity greater than 1, it is easier to work on the Siegel half-space. Recall the map Ψ does this. We are looking for automorphisms of the Siegel half-space that fix $e_{1,\infty}$. We find that these include Heisenberg translations, translations of the form $z + e_1$, and contractions of the second variable (z_1, cz_2) where $|c| \leq 1$.

- If our map ϕ has a triple fixed point at e_1, then we find ϕ is equivalent to the product of a Heisenberg translation and, if not an automorphism, a translation $z + e_1$. If not an automorphism, we obtain the Whole Space/Heisenberg Translation-Translation case.
- The automorphism case will be referred to as the Siegel Half Space/ Heisenberg translation case.

Cases of Multiplicity II.

- If our map ϕ has a double fixed point at e_1, then we have two cases. If both fixed points have the same eigenvalue, ϕ is a translation with characteristic domain the whole space, this will be the Whole Space/Translation case.

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- Finally, our map may be asymptotic to a translation, on the Siegel half space in the form $(z_1 + a, mz_2 + b)$ where $\text{Re}(a) \geq 0$, $|m| \leq 1$, and $|b|^2 \leq (\text{Re}(a))(1 - |m|^2)$. Here the characteristic domain is again the whole space and we have the Whole Space/Asymptotic Translation case.

Model Theory for Linear Fractional Maps in $\mathbb{C}^2$.

We can thus classify linear fractional maps in $\mathbb{C}^2$, resulting in the following seven cases:

- Whole-Space/Dilation
- Half-Space/Dilation
- Siegel Half-Space/Dilation
- Whole-Space/Heisenberg Translation-Translation
- Siegel Half-Space/Heisenberg Translation
- Whole-Space/Translation
- Whole Space/ Asymptotic Translation.

Example.

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Let $\phi(z) = \left(\frac{z_1+3}{4}, \frac{z_2}{2}\right)$. We have

$$m_\phi = \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

where $\lambda_1 = |\lambda_2|^2$ and we thus have a Siegel Half Space/Dilation case.

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where $\lambda_1 = |\lambda_2|^2$ and we thus have a Siegel Half Space/Dilation case. To conform to our model we put $\sigma(z) = e_1 - z$ and $\Phi(z) = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} z$ which gives us

$$\sigma \circ \phi(z) = \left(1 - \frac{z_1+3}{4}, -\frac{z_2}{2}\right) = \left(\frac{1}{4}(1 - z_1), -\frac{1}{2}z_2\right) = \Phi \circ \sigma(z).$$

Composition Operators.

Definition

If ϕ is a self map of the unit ball B_N, the composition operator C_ϕ is defined by

$$C_\phi f = f \circ \phi$$

for f analytic in B_N.

Further Investigations.

Theorem

If ϕ is a linear fractional map of B_N into B_N then C_ϕ is bounded on $H^p(B_N)$ for all $p > 0$. (Cowen and MacCluer 2000)

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If ϕ is an analytic map of B_2 into B_2 then C_ϕ is not necessarily bounded on $H^p(B_N)$ for $p > 0$. In fact, even if ϕ is a polynomial on $H^2(B_2)$ we are not guaranteed boundedness. (J.A. Cima and W.R. Wogen 1987)

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- What can we say about more general analytic maps ϕ from the ball into the ball?

Model Theory for $\mathbb{D}$ (Cowen, 1981).

We can classify analytic maps, ϕ, of $\mathbb{D}$ into $\mathbb{D}$ for which $\phi'(a) \neq 0$ where a is the Denjoy-Wolff point in the following four cases.

- Half-Plane/Dilation
- Half-Plane/Translation
- Plane/Dilation
- Plane/Translation.

(Cowen and MacCluer, 1995).

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