16.9: The Divergence Theorem.

OBJECTIVE

- Introduce the Divergence Theorem.

Recall that we could write Green's theorem in vector form as

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

where C is the positively oriented boundary curve of the plane region D. One might guess that this generalizes to three dimensions as

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{F}(x, y, z)\, dV$$

and one would be correct.

The Divergence Theorem: Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

Thus, the flux of $\mathbf{F}$ along the boundary surface of E is equal to the triple integral of the divergence of $\mathbf{F}$ over E.

Example: Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$, $0 \leq z \leq 1$.

Example: Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} - \frac{1}{2}y^2\,\mathbf{j} + z\,\mathbf{k}$ and the surface consists of three surfaces $z = 4 - 3x^2 - 3y^2$, $1 \leq z \leq 4$ on top, $x^2 + y^2 = 1$, $0 \leq z \leq 1$ on the sides, and $z = 0$ on the bottom.

Homework:

- Section 16.9: 1-13 odds