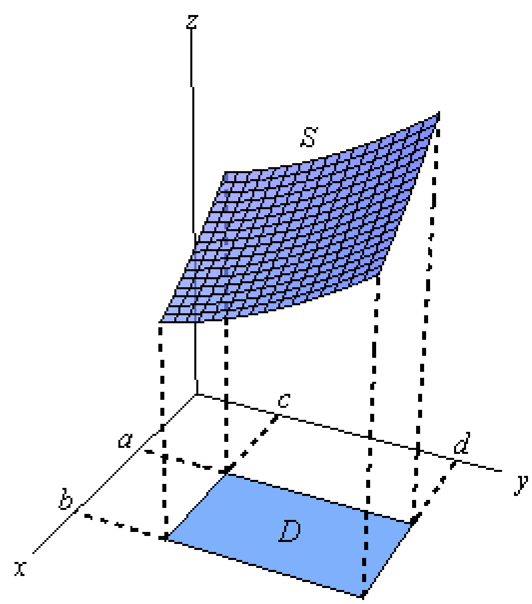
16.7: Surface Integrals.

OBJECTIVE

- Introduce surface integrals of functions and vector fields.

Just as we computed line integrals using arc lengths, we now compute *surface integrals* using surface areas.



We first consider the situation as above, where the surface S is given by $z = g(x,y)$. In such a case, the surface integral is given by

$$\iint_S f(x,y,z)dS = \iint_D f\big(x,y,g(x,y)\big)\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\, dA.$$

What if the surface S is given by $x = g(y, z)$, or $y = g(x, z)$?

What if our surface is piecewise smooth?

More generally, suppose we have the surface parametrization given by:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

Then the surface integral is given by

$$\iint_S f(x, y, z)dS = \iint_D f\big(\mathbf{r}(u, v)\big)|\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

Where D is the range of the parameters as they trace out S.

Example: Evaluate $\iint_S \; 6xy \, dS$ where S is the portion of the plane $x + y + z = 1$ that lies in the first octant and is in front of the yz-plane.

Example: Evaluate $\iint_S \; y + z \, dS$ where S is the surface whose side is the cylinder $x^2 + y^2 = 3$, whose bottom is the disk $x^2 + y^2 \leq 3$ in the xy-plane and whose top is the plane $z = 4 - y$.

Oriented Surfaces and Surface Integrals of Vector Fields:

We start by presuming that we have a two-sided surface (what is an example of a one-sided surface)? We presume our surface has a tangent plane at every point on S except on the boundary. So every point will have two normal vectors $\mathbf{n}_1$and $\mathbf{n}_2 = -\mathbf{n}_1$. Which set we choose gives our surface its orientation.

For a **closed surface**, that is a surface whose boundary consists of some solid region E (think, for example, of a sphere), we say it has **positive orientation** if the normal vectors point away from the solid region E and **negative orientation** if the normal vectors point inward.

For a surface given by the function $z = g(x, y)$, we have that

$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \quad \text{and} \quad \left|\mathbf{r}_x \times \mathbf{r}_y\right| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 + 1}$$

And we associate a natural orientation given by the unit normal vector

$$\mathbf{n} = \frac{-\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial x}\right)^2 + 1}}$$

Where, since $\mathbf{k}$ is positive, we have an *upward* orientation.

For a smooth orientable surface given parametrically by

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

we have a natural orientation given by the unit normal vector defined by

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

Example: Use the parametric representation to find the unit normal vector to a sphere. Is this pointing outward or inward?

Definition: If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S w/ unit normal vector $\mathbf{n}$, then the **surface integral of $\mathbf{F}$ over S** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

This integral is also called the **flux** of $\mathbf{F}$ across S.

We may also write

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

Where D is the parameter domain.

Example: Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the sphere $x^2 + y^2 + z^2 = 1$.

When the surface S is given by $z = g(x, y)$, we may write

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

Example: Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Homework:

- Section 16.7: 3-31 odds