16.6: Parametric Surfaces and Their Areas.

OBJECTIVE

- Discuss Parametric Surfaces, Tangent Planes.
- Find Surface Areas and Surface Areas of Graphs of Functions.

So far our surfaces have all been of special types, cylinders, graphs of functions of two variables, quadric surfaces, etc. In this section we will generalize to more general surfaces, called *parametric surfaces*.

Similar to how we described a space curve as a vector function $\mathbf{r}(t)$ of one variable, we may describe a surface as a vector function of two variables:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

where $\mathbf{r}(u, v)$ is a vector-valued function defined on a region D in the uv-plane. Such a parametrization is called the *parametric representation* of the *parametric surface* S.

The parametric equations of a surface S are given by

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

Example: Determine the surface given by the parametric representation

$$\mathbf{r}(u, v) = u\mathbf{i} + u\cos v\ \mathbf{j} + u\sin v\,\mathbf{k}$$

What happens if we hold u constant? What about v?

Example: Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and that contains two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$.

Example: Find a parametric representation of the following surfaces:

a) The elliptic paraboloid $y = 3x^2 + 2z^2 - 6$.

b) The sphere $x^2 + y^2 + z^2 = 4$.

c) The cylinder $y^2 + z^2 = 16$.

Tangent Planes: Suppose we have a parametric surface S given by

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

We may define the following:

$$\mathbf{r}_u(u,v) = \frac{\partial x}{\partial u}(u,v)\mathbf{i} + \frac{\partial y}{\partial u}(u,v)\mathbf{j} + \frac{\partial z}{\partial u}(u,v)\mathbf{k}$$

$$\mathbf{r}_v(u,v) = \frac{\partial x}{\partial v}(u,v)\mathbf{i} + \frac{\partial y}{\partial v}(u,v)\mathbf{j} + \frac{\partial z}{\partial v}(u,v)\mathbf{k}$$

If we hold $v = v_0$ fixed, then $\mathbf{r}_u(u, v_0)$ will be tangent to the curve $\mathbf{r}(u, v_0)$, as long as $\mathbf{r}_u(u, v_0) \neq 0$. Likewise for $u = u_0$. Thus, provided neither is zero, we have that $\mathbf{r}_u(u_0, v_0)$ and $\mathbf{r}_v(u_0, v_0)$ will be tangent to the surface S given by $\mathbf{r}(u, v)$ at (u_0, v_0). Additionally, the tangent plane to the surface at this point will contain both $\mathbf{r}_u(u_0, v_0)$ and $\mathbf{r}_v(u_0, v_0)$. Thus, as long as $\mathbf{r}_u \times \mathbf{r}_v \neq 0$, this vector will be ortogonal to the surface S at (u_0, v_0) and thus can be used to determine the tangent plane.

Example: Find the equation of the tangent plane to the surface $x = u^2 + 1, y = v^3 + 1, z = u + v$ at the point (5,2,3).

Surface Area: To find the surface area of a parametric surface S, use

Definition: If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad (u, v) \in D$$

And S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

Where $\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$ and $\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$.

Example: Find the surface area of the portion of the sphere of radius 4 that lies inside the cylinder $x^2 + y^2 = 12$ and above the xy-plane.

Surface area of a function: For the case where we can write the surface S with equations $z = f(x, y)$ where (x, y) lies in our domain D and f has continuous partial derivatives, we have parametric equations

$$x = x \qquad y = y \qquad z = f(x, y)$$

And may derive the equation

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$$

Example: Find the area of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Homework:

- Section 16.6: 1-5, 13-25, 33-35, 39-49 odds