16.5: Curl and Divergence.

OBJECTIVE

- Discuss Curl and Divergence.

In this section we discuss two operations that can be performed on vector fields.

Curl: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field defined by

$$\text{curl } \mathbf{F} = \left(\frac{\partial \mathrm{R}}{\partial \mathrm{y}} - \frac{\partial \mathrm{Q}}{\partial \mathrm{z}}\right)\mathbf{i} + \left(\frac{\partial \mathrm{P}}{\partial \mathrm{z}} - \frac{\partial \mathrm{R}}{\partial \mathrm{x}}\right)\mathbf{j} + \left(\frac{\partial \mathrm{Q}}{\partial \mathrm{x}} - \frac{\partial \mathrm{P}}{\partial \mathrm{y}}\right)\mathbf{k}$$

How do I memorize this? Define the following vector operator:

$$\nabla = \mathbf{i}\frac{\partial}{\partial \mathrm{x}} + \mathbf{j}\frac{\partial}{\partial \mathrm{y}} + \mathbf{k}\frac{\partial}{\partial \mathrm{z}}$$

Then what is $\nabla \times \mathbf{F}$?

So we have

$$\nabla \times \mathbf{F} = \text{curl } \mathbf{F}$$

Example: Find the curl of $\mathbf{F}(x, y, z) = \ln(2y + 3z)\,\mathbf{i} + \ln(x + 3z)\,\mathbf{j} + \ln(x + 2y)\,\mathbf{k}$.

Fact: If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl } \nabla f = \mathbf{0}$$

Corollary: If $\mathbf{F}$ is conservative, then curl $= \mathbf{0}$.

Was the vector field in the above example conservative?

Theorem: If $\mathbf{F}$ is a vector field defined on all of R^3 whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

Example: Determine whether or not $\mathbf{F}(x, y, z) = 2xyz^4\mathbf{i} + x^2z^4\mathbf{j} + 4x^2yz^3\mathbf{k}$ is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

Divergence: If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field and $\frac{\partial \mathrm{P}}{\partial \mathrm{x}}, \frac{\partial \mathrm{Q}}{\partial \mathrm{y}}, \frac{\partial \mathrm{R}}{\partial \mathrm{z}}$ all exist, then the divergence of $\mathbf{F}$ is the function of three variables defined by

$$\text{div } \mathbf{F} = \frac{\partial \mathrm{P}}{\partial \mathrm{x}} + \frac{\partial \mathrm{Q}}{\partial \mathrm{y}} + \frac{\partial \mathrm{R}}{\partial \mathrm{z}}$$

In terms of the gradient, we may write

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

Example: Find the divergence of $\mathbf{F}$ if $\mathbf{F}(x, y, z) = x^2y\,\mathbf{i} + xyz\,\mathbf{j} + x^2y^{2019}\mathbf{k}$.

Note that if $\mathbf{F}$ is a vector field, then so is curl $\mathbf{F}$, so it makes sense to compute its divergence. Using the definitions, If $\mathbf{F}$ is a vector field defined on all of R^3 whose component functions have continuous second-order partial derivatives one can show

$$\text{div curl } \mathbf{F} = 0$$

The Laplace Operator: If f is a function of three variables, we have

$$\text{div } \nabla f = \nabla \cdot (\nabla \text{f}) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

where, due to its high frequency in applications, is often abbreviated by $\nabla^2 f$ where the operator $\nabla^2 = \nabla \cdot \nabla$ is called the **Laplace operator** because of its relation to Laplace's equation

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Vector Forms of Green's Theorem: If we suppose all of the conditions for Green's theorem hold, we may rewrite Green's theorem as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA$$

Or as

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \text{div } \mathbf{F}(x, y) \, dA$$

Homework:

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