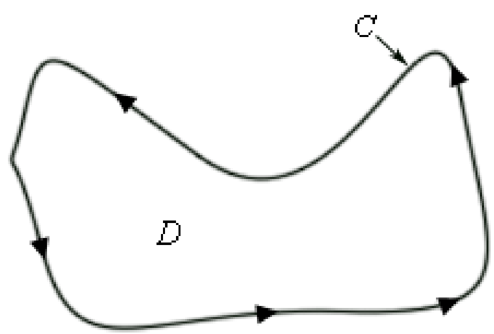
16.4: Green's Theorem.

OBJECTIVE

- Introduce Green's Theorem.

In this section we will look at a relationship between certain line integrals around a simple closed curve and double integrals.

We began with a simple closed curve C and the region D enclosed by the curve.



We take the convention that the **positive orientation** of a simple closed curve C refers to a single counterclockwise traversal of C. In such a case, D is to the *left* of C.

Given such a curve/region, we have the following theorem.

Green's Theorem: Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Other notation used is $\oint_C$ and $\int_{\partial \mathrm{D}}$.

Example: Use Green's theorem to evaluate $\oint_C xydx + x^2y^3dy$ where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$ with positive orientation.

Sometimes the line integral can help us compute the double integral as well. What was the area formula for a region D in the plane?

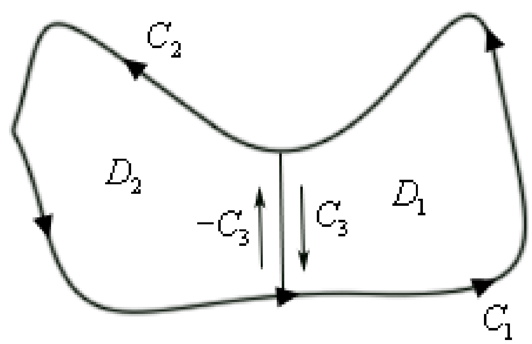
Using Green's theorem, what does this tell us about $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$? What are some examples of this?

Example: Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Extended version of Green's Theorem: Suppose D is the region $D = D_1 \cup D_2$ as shown below, where D_1 and D_2 are simple. Then we have

$$\int_{C_1 \cup C_3} P dx + Q dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int_{C_2 \cup (-C_3)} P dx + Q dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



Adding along these two equations, the line integrals along C_3 and $-C_3$ cancel. We obtain

$$\int_{C_1 \cup C_2} P dx + Q dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

We can do this for any finite union of non-overlapping simple regions.

Example: Evaluate $\oint_C \; y^3dx - x^3dy$ where C comprises of the two circles of radius 1 and radius 2 centered at the origin with positive orientation.

Homework:

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