16.3: The Fundamental Theorem for Line Integrals.

OBJECTIVE

- Introduce the fundamental theorem for line integrals.
- Discuss conservative vector fields and independence of paths.

Recall the fundamental theorem of calculus:

If we think of the gradient vector ∇f as a sort of derivative of f, then we may consider the following theorem a fundamental theorem for line integrals

The Fundamental Theorem of Line Integrals: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$**,** $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Independence of Path: Notice that the above theorem only relied on the endpoints. We could have taken *any* path from the initial point to the terminal point. We see that for a *conservative vector field*, the integral only depends on the endpoints. We say that line integrals of conservative vector fields are independent of path.

A curve is said to be *closed* if its initial and terminal points coincide, that is $\mathbf{r}(b) = \mathbf{r}(a)$. In such a case, one can show the following

> **Theorem:** $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in a domain D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

In the following theorem, we assume our domain D is **open** and **connected**.

> **Theorem:** Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

We are still left with the question, how do we know when $\mathbf{F}$ is a conservative vector field?

Theorem: If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Show this!

For a simply-connected domain (what's that?) we have a converse statement.

Theorem: Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field on an open simpy-connected region D. Suppose that P and Q have continuous first-order partial derivatives and throughout D

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Then $\mathbf{F}$ is conservative.

Example: Determine whether or not the vector field $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ is a conservative vector field.

Example: Find a function f such that $\mathbf{F} = \nabla f$ where $\mathbf{F}(x, y) = (3 + 2xy^2\mathbf{i} + 2yx^2\mathbf{j}$ and use this to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C where C is the arc of the hyperbola $y = \frac{1}{x}$ from $(1,1)$ to $(4, \frac{1}{4})$.

Homework:

- Section 16.3: 3-19, 29-33 odds