# 16.2: Line Integrals.

***OBJECTIVE***

- Introduce Line Integrals with Respect to Arc Length.
- Introduce Line Integrals with Respect to $x$ and $y$.
- Introduce Line Integrals in Space.
- Introduce Line Integrals of Vector Fields.

In this section we generalize the idea of integrating a function over an interval $[a, b]$, where we see $x$ realize all values in the intervals, to integrating over a curve $C$. These integrals are called **line integrals**. We will integrate a function $f(x, y)$ of two variables. This will be different than double integration as the points $(x, y)$ will lie on the curve $C$ in the $xy$-plane, not a two-dimensional region.

We start with a plane curve $C$ given by the parametric equations

$$x = x(t) \qquad \text{and} \qquad y = y(t) \qquad a \leq t \leq b$$

Or, equivalently, by the vector equation $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. We assume that our curve is *smooth*, meaning that $\vec{r}'(t)$ is continuous and not zero.

For a smooth curve $C$, we write the **line integral of $f$ along $C$** by

$$\int_C f(x,y)ds$$

If the limit exists, where $ds$ is used to remind us that we are integrating along a curve. In particular, recall that the arc length of a curve is given by

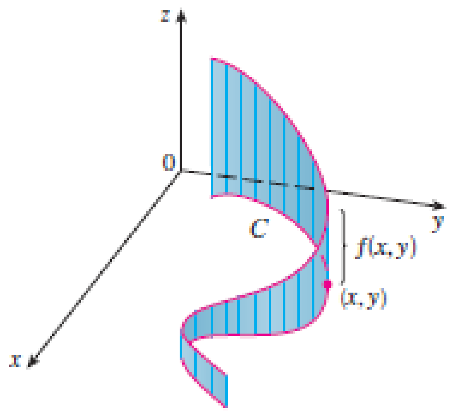
$$L = \int_a^b ds \quad \text{where} \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Here, it turns out $ds$ is the same quantity as above. Thus we have

$$\int_C f(x,y)ds = \int_a^b f\big(x(t), y(t)\big)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

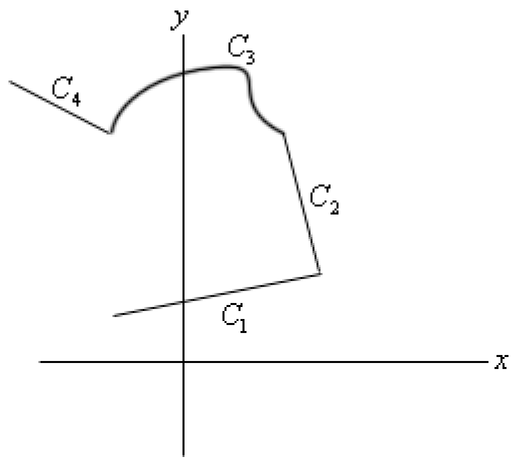
$$= \int_a^b f\big(\vec{r}(t)\big)|\vec{r}'(t)|dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as $t$ goes from $a$ to $b$.




**Example:** Evaluate $\int_C xy^4 ds$ where $C$ is the right half of the circle $x^2 + y^2 = 16$ traced out in a counterclockwise direction.

**Piecewise Smooth Curves:** A piecewise smooth curve is any curve that can be written as the union of a finite number of smooth curves.




For the image above, we have

$$\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + \int_{C_3} f(x,y)ds + \int_{C_4} f(x,y)ds$$

**Example:** Evaluate $\int_C 2x\, ds$ where $C$ consists of the arc $C_1$ of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the line segment $C_2$ from $(1,1)$ to $(2,3)$.

What if we just evaluated $\int_C 2xds$ along the line segment from $(2,3)$ to $(1,1)$?

**Line Integrals with Respect to $x$ and $y$:** Again, starting with a plane curve $C$ given by the parametric equations

$$x = x(t) \qquad \text{and} \qquad y = y(t) \qquad a \leq t \leq b$$

We say the **line integral with respect to $x$** is

$$\int_C f(x,y)dx = \int_a^b f\big(x(t),y(t)\big)\, x'(t)dt$$

And the **line integral with respect to $y$** is

$$\int_C f(x,y)dy = \int_a^b f\big(x(t),y(t)\big)\, y'(t)dt$$

These frequently occur together. We abbreviate by writing

$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \int_C P(x,y)dx + Q(x,y)dy$$

**Example:** Evaluate $\int_C y^2\, dx + xdy$ where $C$ is the arc of the parabola $x = 4 - y^2$ from $(0, -2)$ to $(0,2)$.

What if we just took the shortcut straight up the $y$ –axis?

**Line Integrals in Space:** We now start with a *space* curve $C$ given by the parametric equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t) \qquad a \leq t \leq b$$

Or, equivalently, by the vector equation $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$ We again assume that our curve is *smooth*.

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_a^b f\big(\vec{r}(t)\big)|\vec{r}'(t)|dt$$

We also write

$$\int_C P(x,y,z)dx + \int_C Q(x,y,z)dy + \int_C R(x,y,z)dz$$

$$= \int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

**Example:** Evaluate $\int_C \ y\,dx + xdy + zdz$ where $C$ is given by $x = \cos t$ $y = \sin t, z = t^2, 0 \le t \le 2\pi$.

**Line Integrals of Vector Fields:** Above we saw line integrals of functions. Now we look at line integrals of vector fields. Suppose we have the continuous vector field $\vec{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ and smooth curve $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},\ a \le t \le b$. We define the **line integral of F along $C$** by

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)dt = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$$

**Example:** Evaluate $\int_C \mathbf{F} \cdot d\boldsymbol{r}$ where $\mathbf{F}(x, y, z) = 8x^2yz\,\mathbf{i} + 5z\,\mathbf{j} - 4xy\,\mathbf{k}$ and $C$ is the curve given by $\vec{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \leq t \leq 1$.

**Facts:** One can show the following:

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} = \int_C P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$$

and

$$\int_{-C} \mathbf{F} \cdot d\boldsymbol{r} = -\int_C \mathbf{F} \cdot d\boldsymbol{r}$$

**Homework:**

- Section 16.2: 1-21, 29(a) odds