15.9: Change of Variables in Multiple Integrals.

OBJECTIVE

- Introduce the Jacobian.
- Introduce the Change of Variables Equations.

Suppose you have to integrate $\int_0^1 x\sqrt{2-x^2}dx$. What method would you use?

What does this rule look like in general?

This “change of variables” formula is what you have been implicitly using when applying the substitution rule. We have already seen this applied to double integrals in one case (what is it?). How do we generalize this?

We began with the following definition.

Definition: The *Jacobian* of the transformation given by $x = x(u,v)$ and $y = y(u,v)$ is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

and the *Jacobian* of the transformation given by $x = x(u,v,w)$, $y = y(u,v,w)$, and $z = z(u,v,w)$ is

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Below we assume that the functions are continuously differentiable and behave well on the boundaries.

Change of Variables Formula for Multiple Integrals:

Let $x = x(u, v)$ and $y = y(u, v)$ define a one-to-one mapping of a region T in the uv-plane onto a region R in the xy-plane such that the Jacobian is never in T. Then

$$\iint_R f(x, y)dA(x, y) = \iint_T f(x(u, v), y(u, v))|J(u, v)|dA(u, v)$$

where $dA(x, y)$ and $dA(u, v)$ denote the area elements of (x, y) and (u, v), respectively.

Likewise, if $x = x(u, v, w)$, $y = y(u, v, w)$ and $z = z(u, v, w)$ define a one-to-one mapping of a solid D in the uvw-space onto a solid S in the xyz-space such that the Jacobian is never in D. Then

$$\iiint_S f(x, y, z)dV(x, y, z) = \iiint_D f(x(u, v, w), y(u, v, w), z(u, v, w))|J(u, v, w)|dV(u, v, w)$$

Example: Compute the Jacobian for the change of variables from Cartesian to polar coordinates.

Example: Evaluate $\iint_R \; e^{\frac{x-y}{x+y}} dA$ where

$$R = \{(x, y) \mid x \geq 0, y \; \geq 0, x + y \leq 1\}.$$

Homework:

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