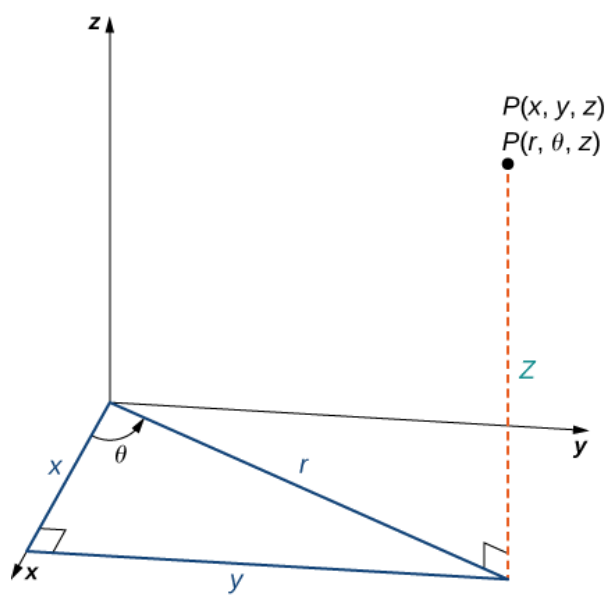
15.7-15.8: Triple Integrals in Cylindrical and Spherical Coordinates.

OBJECTIVE

- Introduce Cylindrical Coordinates.
- Introduce Spherical Coordinates.

Just as we saw that polar coordinates are often a convenient description of plane geometry, we will take a look at two alternative coordinate systems in three dimensions.

Cylindrical Coordinates: We began with the cylindrical coordinate system. This is simply an extension of polar coordinates into three dimensions.



Conversion Equations from Cylindrical to Cartesian Coordinates:
To convert from cylindrical to Cartesian coordinates, use the equations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Conversion Equations from Cartesian to Cylindrical Coordinates:
To convert from cylindrical to Cartesian coordinates, use the equations

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

For a constant c, what do the equations $x = c$, $y = c$, and $z = c$ look like?

What about the equations $r = c$, $\theta = c$, and $z = c$?

Example: Plot the point given by Cylindrical coordinates $(2, -\frac{\pi}{2}, 1)$. Find the rectangular coordinates of the point as well.

Example: Change $(-2, 2\sqrt{3}, 3)$ from rectangular to cylindrical coordinates.

Example: Identify the surfaces $r^2 + z^2 = 9$ and $r = \cos\theta$.

Evaluating triple integrals with cylindrical coordinates: Combining what we know about evaluating triple integrals and evaluating double integrals in polar coordinates, for a type I region E whose projection D onto the plane is conveniently described by polar coordinates so that

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

And D is given in polar coordinates by

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

we have, for f continuous, the following

$$\iiint_E f(x, y, z) dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r \, dz dr d\theta$$

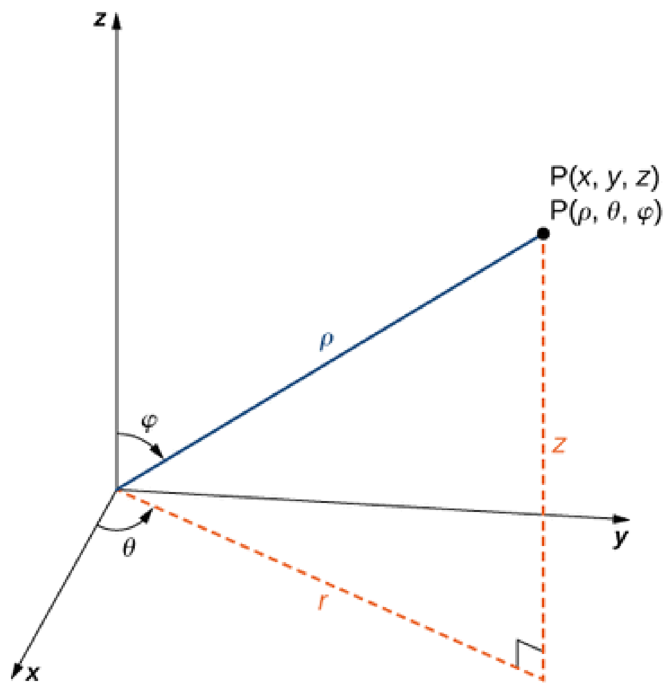
Example: Evaluate $\iiint_E (x - y)dV$ where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$, above the xy-plane and below the plane $z = y + 4$.

Example: Find the volume of the solid that lies between the paraboloid $x = y^2 + z^2$ and the sphere $x^2 + y^2 + z^2 = 2$

Homework:

- Section 15.7: 1-29 odds

Spherical Coordinates: We saw that cylindrical coordinates were convenient when we had symmetry about an axis. For triple integrals evaluated around a sphere or cone, we have another coordinate system that is more convenient.



The spherical coordinates of a point $P(\rho, \theta, \varphi)$ are shown above where ρ is the distance from the origin to the point P, θ is the same angle as in cylindrical coordinates, and φ is the angle between the positive z-axis and the line segment created by OP where O is the origin. In particular we have

$\rho \geq 0$ and $0 \leq \varphi \leq \pi$

From the above picture we can deduce the following conversions:

Conversion Equations from Spherical to Cartesian Coordinates:
To convert from spherical to Cartesian coordinates, use the equations

$$x = \rho \sin \varphi \cos \theta \qquad y = \rho \sin \varphi \sin \theta \qquad z = \rho \cos \varphi$$

Conversion Equations from Cartesian to Spherical Coordinates:
The distance formula gives

$$\rho^2 = x^2 + y^2 + z^2$$

Example:

a) Convert the point $(\sqrt{6}, \frac{\pi}{4}, \sqrt{2},)$ from cylindrical to spherical coordinates. Plot the point.

b) Convert the point $(-1, 1, -\sqrt{2},)$ from Cartesian to spherical coordinates. Plot the point.

Example: Identify the surfaces $\varphi = \frac{\pi}{3}$ and $\rho = \sin\theta \sin\varphi$.

Evaluating triple integrals with spherical coordinates: The counterpart of a rectangular box for spherical coordinates is given by a *spherical wedge*

$$E = \{(\rho, \theta, \varphi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \varphi \leq d\}$$

where $a \geq 0$ and $\beta - \alpha \leq 2\pi$, and $d - c \leq \pi$.

For f continuous, one find the following formula for triple integration

$$\iiint_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)\rho^2 \sin\varphi \, d\rho d\theta d\varphi$$

Caution! Just as with polar coordinates, do not forgot to convert the differential $\rho^2 \sin\varphi$ as well.

Example: Evaluate $\iiint_E 16z\, dV$ where E is the upper half of the sphere $x^2 + y^2 + z^2 = 4$.

Example: Find the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$ using *spherical coordinates.*

Homework:

- Section 15.8: 1-37 odds