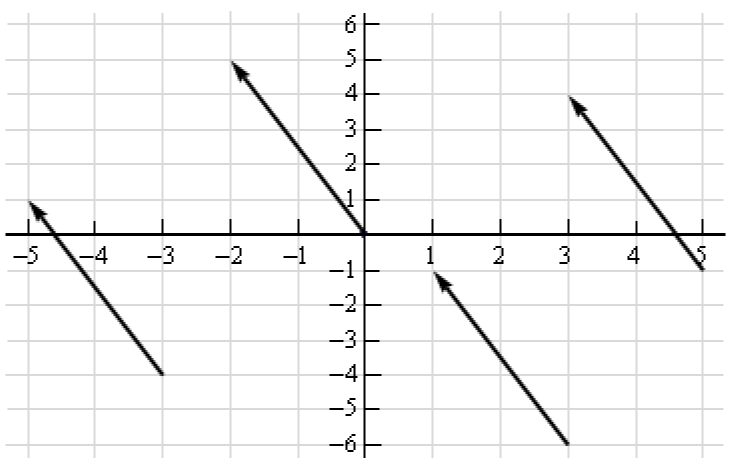
Chapter 12 Review:

OBJECTIVE

- Review vectors, dot product, cross product, and equations of lines and planes.

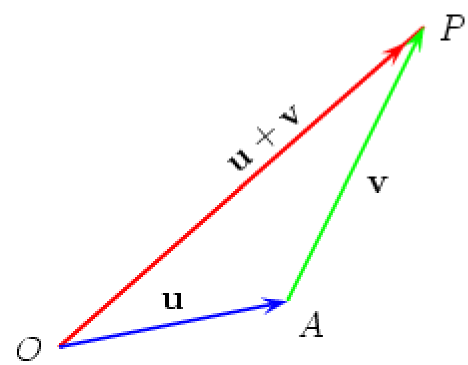
Recall that a **vector** is a quantity that has both magnitude *and* direction. It is often represented by an arrow or directed line segment. The length of the line segment signifies its magnitude and the direction of the line segment is the direction of the vector. Vectors, as opposed to scalars which only have magnitude, are represented by letters in **boldface** (or, if you just have a pencil, by putting an arrow on top of the letter).

Because vectors don't give any information about where the quantity is, any directed line segment with the same length and direction will represent the same vector. See below.



In the diagram above, each arrow moves two units to the left and five units up. We notate this as $\vec{v} = \langle -2,5 \rangle$. Each vector above is equivalent. The zero vector is the only vector with no specific direction.

The Arithmetic of Vectors: To add two vectors **u** and **v**, relocate v so that the tail coincides with the tip of **u** and define the sum **u+v** to be the vector from the initial point of **u** to the terminal point of **v**. See below.



This is sometimes called the triangle law for obvious reasons.

- What if we started with **v** and added **u** instead? Draw a picture.

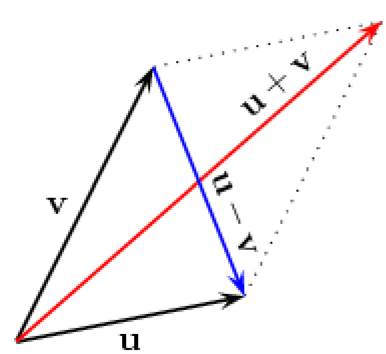
This is sometimes called the parallelogram law.

Whereas vectors have magnitude and direction, *scalars* have only magnitude. With this in mind, we can define *multiplication of a vector by a scalar*.

Definition: If c is a scalar and $\vec{v}$ is a vector, then the scalar multiple $c\vec{v}$ is the vector whose length is $|c|$ times the length of the vector $\vec{v}$ and whose direction is the same as $\vec{v}$ if c is positive and of opposite direction if c is negative.

Note that two nonzero vectors are *parallel* if they are scalar multiples of each other.

Subtraction of two vector **u** and **v** is just a special case of addition where the second vector **v** multiplied by the scalar -1. Thus **u+v=u+(-v)**.



Example: Find the sum of the given vectors and illustrate them geometrically: $\vec{a} = \langle 5,0 \rangle$, $\vec{b} = \langle -1,2 \rangle$

We can handle vectors algebraically by imposing them on a Cartesian coordinate system. In two-dimensions, if we are given a vector $\vec{v} = \langle v_1, v_2 \rangle$ and place its initial point at the origin, its terminal point has coordinates (v_1, v_2). Likewise, in three-dimensions, if we are given a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and place its initial point at the origin, its terminal point has coordinates (v_1, v_2, v_3). The coordinates are called *components of* $\vec{v}$. Note the difference in notation.

Given two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\vec{a}$ with representation $\overrightarrow{AB}$ is given by

$$\vec{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

Example: Find the vector represented by the directed line segment with initial point $A(4,0,-2)$ and terminal point $B(1,5,-1)$.

The **magnitude** or **length** of a 2D vector $\vec{a} = \langle a_1, a_2 \rangle$ is given by

$$|\vec{v}| = \sqrt{{a_1}^2 + {a_2}^2}\ .$$

The **magnitude** or **length** of a 3D vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is given by

$$|\vec{v}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}\ .$$

If $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$ then

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \quad \text{and} \quad \vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

And $c\vec{a} = \langle ca_1, ca_2 \rangle$.

- Can you guess how to define addition and subtraction in 3D? What about in n dimensions?

Example: If $\vec{a} = \langle 2,-1,4 \rangle$ and $\vec{b} = \langle 1,-1,5 \rangle$, find $\vec{a} - \vec{b}$, $2\vec{a} + 4\vec{b}$, $|\vec{a}|$, and $|\vec{a} - \vec{b}|$.

Properties of Vectors: Let **a**, **b**, **c**, be vectors and *c, d* be scalars, then

1. **a+ b=b+a**
2. **a+(b+c)=(a+b)+c**
3. **a+0=a**
4. **a+(-a)=0**
5. *c* **(a+b)=** *c* **a+** *c* **b**
6. (*c*+*d*) **a=** *c* **a+** *d* **a**
7. (*c d*) **a=***c*(*d* **a**)
8. 1 **a=a**

The Standard Basis Vectors:

There are three simple vectors that are so useful that they are given their own notation. These are the standard basis vectors:

In 2D we have $\mathbf{i} = \langle 1, 0 \rangle$, $\mathbf{j} = \langle 0, 1 \rangle$,

And in 3D we have $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.

Not only are these vectors really simple, it turns out we can write every vector as a linear combination of these.

- Show this.

Unit Vectors: A unit vector is simply a vector with length 1. Our standard basis vectors above are unit vectors. For $\vec{a} \neq 0$, the unit vector that has the same direction as $\vec{a}$ is given by

$$\vec{u} = \frac{1}{|\vec{a}|}\vec{a} = \frac{\vec{a}}{|\vec{a}|}.$$

- Verify this.

Example: Find a unit vector that has the same direction as $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Example: If $\vec{a} = 2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$ and $\vec{b} = \mathbf{i} + 3\mathbf{j}$, find $\vec{a} - \vec{b}$, $2\vec{a} + 4\vec{b}$, $|\vec{a}|$, and $|\vec{a} - \vec{b}|$.

Above we defined multiplication of a scalar by a vector. We next recall *two* ways to define multiplication of a vector by a vector. We began with our first definition.

Definition: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ then the **dot product** of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$, is given by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Caution! When taking the dot product of two vectors, the result is a *scalar*! The dot product is also known as the scalar product or inner product.

The definition for n-dimensions is similar.

Example: Find $\vec{a} \cdot \vec{b}$ if $\vec{a} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ and $\vec{b} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

Example: What is the dot product for the standard basis vectors?

Properties of the Dot Product: If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and *c* is a scalar, then

1. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

2. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

3. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$

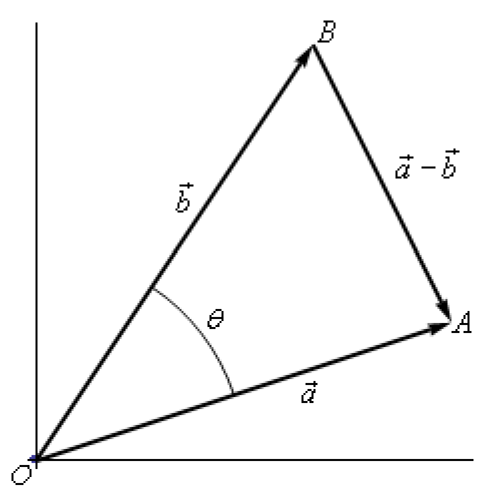
4. $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) == \vec{a}\cdot(c\vec{b})$

5. $\vec{0}\cdot\vec{a} = 0$

Geometric Interpretation of the Dot Product:

Theorem: If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$ then

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta \quad \text{and} \quad \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$



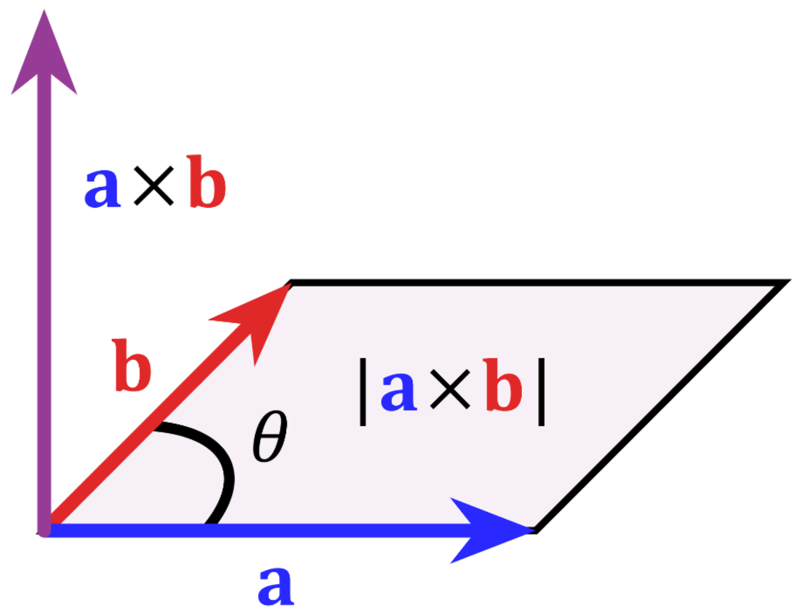
Example: Find the angle between the vectors $\vec{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\vec{b} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

- When are two vectors orthogonal?

- Parallel?

In the previous lesson, we defined multiplication of a *vector* by a *vector* to obtain a *scalar*. This definition extended easily to n-dimensions. We now introduce an alternate definition of for multiplying two *vectors* that gives us a *vector*. This will be called the **cross product,** also called the **vector product** and only applies for 3D vectors.

Our motivation will be to define the product of two vectors such that our resulting vector is *perpendicular* to our original two vectors.



Definition: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ then the **cross product** of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$, is given by

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

If you are concerned about remembering this formula, there is an easier way to remember it. We introduce **Determinants** of order 2 and 3. While there is a precise way to define Determinants of any order, all we need for now are the following results:

DETERMINANTS

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A **determinant of order 3** can be defined in terms of second order determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

With this we can write the cross product in a way that is easier to remember.

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: Find $\vec{a} \times \vec{b}$ if $\vec{a} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ and $\vec{b} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

Example: Find $\vec{a} \times \vec{a}$ if $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

Example: Find $(\vec{a} \times \vec{b}) \cdot \vec{a}$ and $(\vec{a} \times \vec{b}) \cdot \vec{b}$.

- What does this tell us about the vector $\vec{a} \times \vec{b}$? (hint: what does the dot product result above tell us?)

If we represent $\vec{a}$ and $\vec{b}$ by directed line segments with the same initial point, then we can use the Right-Hand Rule again to determine the direction of $\vec{a} \times \vec{b}$.

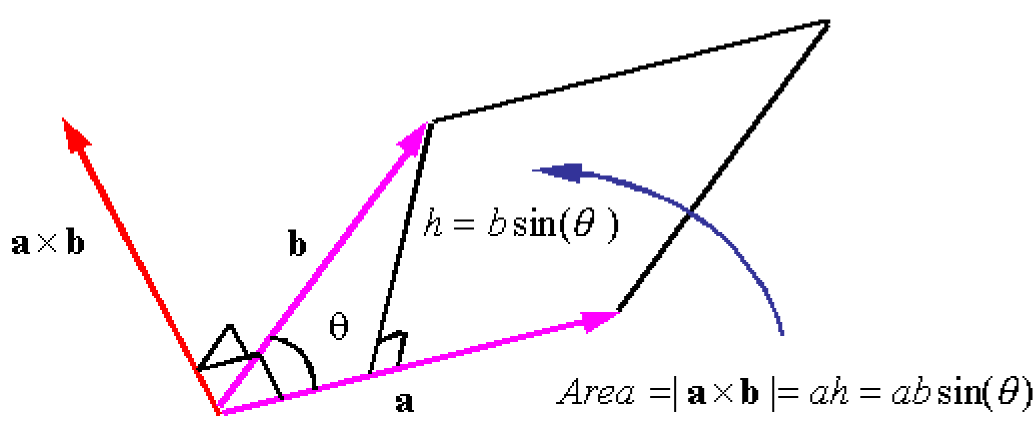
Theorem: If θ is the angle between $\vec{a}$ and $\vec{b}$ such that $0 \le \theta \le \pi$, then

$$\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right| \sin\theta .$$

Corollary: Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if

$$\vec{a} \times \vec{b} = 0 .$$

We can also find a geometric interpretation for the cross product:



From above it is clear that the length of the cross product $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

Example: Just as a line is determined by two points, a plane is determined by three points. If a plane contains the points P=(1,0,0), Q=(1,1,1) and R=(2,-1,3), find a vector that is orthogonal to the plane.

Example: Find the cross product for the standard basis vectors.

- Is the cross product commutative?

Example: Compute $\mathbf{i} \times (\mathbf{i} \times \mathbf{j})$ and $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$. Use the above results to simplify your work

- What can you conclude above the associative law for multiplication with respect to the cross product?

While we lose some of the usual algebra laws, we do retain many of them.

Properties of the Cross Product: If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and *c* is a scalar, then

6. $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

7. $(c\vec{a})\times\vec{b} = c(\vec{a}\times\vec{b}) == \vec{a}\times(c\vec{b})$

8. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

9. $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}+\vec{b}\times\vec{c}$

10. $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$

11. $\vec{a}\times(\vec{b}\times\vec{c}) = \left(\vec{a}\cdot\vec{c}\right)\vec{b} - \left(\vec{a}\cdot\vec{b}\right)\vec{c}$

- Which of these are scalars? Which are vectors?

Example: Determine if the three vectors $\vec{a} = \langle 1,4,-7 \rangle$, $\vec{b} = \langle 2,-1,4 \rangle$, and $\vec{c} = \langle 0,-9,18 \rangle$ lie in the same plane or not.

Equations of Lines in 3D:

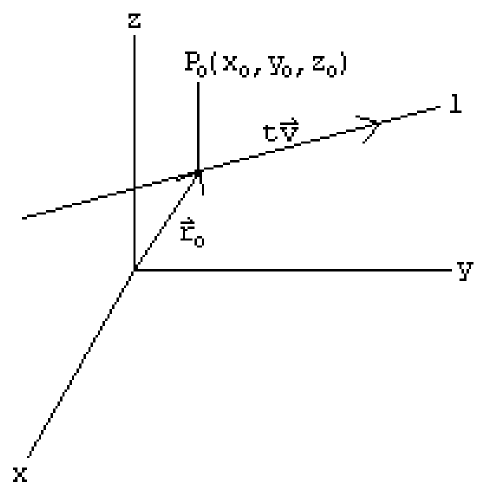
In 2D, recall the general equation of a line is _________.

- What does this look like in 3D?

In 2D, we need the slope of our line (to determine its angle of inclination) and a point (to translate it to the correct place in our space).

In 3D, we recall that we can use a vector to determine the direction of a line. We then need a point to translate it to the correct place in our space.

Our program is then as follows: we want to describe a line L in 3D. We let $\vec{v}$ be a vector parallel to L and $P(x, y, z)$ be an arbitrary point on our line L.



Let $\overrightarrow{r_0}$ and $\vec{r}$ be the position vectors of P_0 and P. We can use the triangle law for vectors to find $\vec{r} - \vec{r}_0 = \vec{a}$. Now, $\vec{a}$ and $\vec{v}$ are parallel vectors, thus we can write $\vec{a} = t\vec{v}$ and we obtain the equation

$$\vec{r} = \vec{r}_0 + t\vec{v}.$$

This is a vector equation of L. For each value of the parameter *t*, we obtain the position vector $\vec{r}$ of a point on L.

We see from this that the positive values of *t* correspond to the points that lie on one side of P_0 and negative values of *t* correspond to points that lie on the other side. As the *t* values vary, our line L is traced out.

If $\vec{v} = \langle a, b, c \rangle$ is a vector that gives the direction of our line, and $\vec{r} = \langle x, y, z \rangle$ and $\overrightarrow{r_0} = \langle x_0, y_0, z_0 \rangle$, then we can write our vector equation in component form as

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle.$$

Since two vectors are equal if and only if their components are equal, we obtain the following parametric equations for a line through the point (x_0, y_0, z_0) parallel to the direction vector $\langle a, b, c \rangle$:

$$x = x_0 + at \qquad\qquad y = y_0 + bt \qquad\qquad z = z_0 + ct$$

where each value of *t* gives a point on the line L.

If we presume each component is nonzero, we can solve the above equations for *t* to obtain the following equations, known as the **symmetric equations** of L:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Example: Find the vector, parametric, and symmetric equations for the line through the point (1,0,-3) and parallel to the vector $\langle 2,-4,5 \rangle$.

Example: Find the parametric and symmetric equations of the line through the points (1,2,0) and (-5,4,2).

To represent only a line segment, we use the following results:

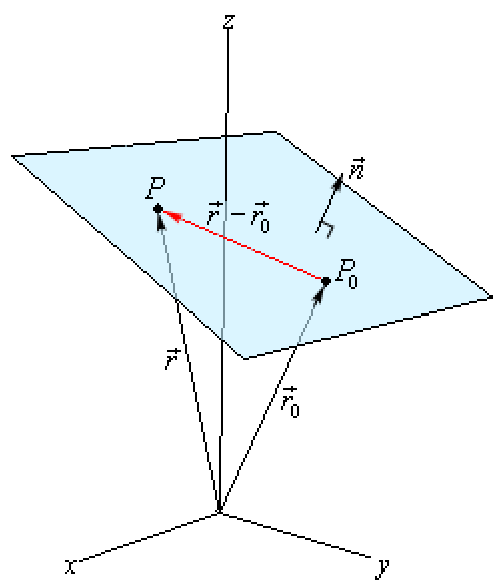
- The line segment from $\vec{r_0}$ to $\vec{r_1}$ is given by the vector equation

$$\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1} \qquad 0 \le t \le 1.$$

Equations of Planes in 3D:

To describe a plane in space, we can completely specify its direction using a vector that is **perpendicular** to the plane. Thus we can determine a plane with a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\vec{n} = \langle a, b, c \rangle$ that is normal (that is, perpendicular) to the plane. This is called a *normal vector*.

Assume $P(x, y, z)$ is any point on our plane. Let and $\vec{r_0}$ and $\vec{r}$ be the position vectors for $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$, respectively.



From above we see that the vector $\vec{r} - \vec{r_0}$ lies completely in our plane. For convenience, our normal vector is pictured on the plane. However, it does not have to be. Our normal vector is orthogonal to any vector that lies in the plane and is thus orthogonal to the vector $\vec{r} - \vec{r_0}$. Thus we have

$$\vec{n} \cdot \left(\vec{r} - \vec{r_0}\right) = 0 \text{ or } \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}.$$

Either equation is known as the **vector equation of the plane**.

If we actually compute the dot product of the components, we obtain the **scalar equation of the plane**.

If our plane goes through the point $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$, the scalar equation of the plane is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Rearranging the above equation, we can also write the equation of a plane as

$$ax + by + cz + d = 0.$$

- Do this.

Example: Determine the equation of the plane that contains the points P(1,-2,0), Q(3,1,4), and R(0,-1,2).

Example: Determine if the plane given by $-x+2z=10$ and the line given by $\vec{r}=\langle 5,2-t,10+4t\rangle$ are orthogonal, parallel, or neither.

Homework:

- Section 12.2: 3-27 odds
- Section 12.3: 1-27 odds
- Section 12.4: 1-19 odds
- Section 12.5: 3-43 odds