15.6: Triple Integrals

OBJECTIVE

- Introduce the Triple Integral.
- Applications of the Triple Integral.

We define a triple integral by extending the notions we used to define a double integral. We began with the case when f is defined on a box

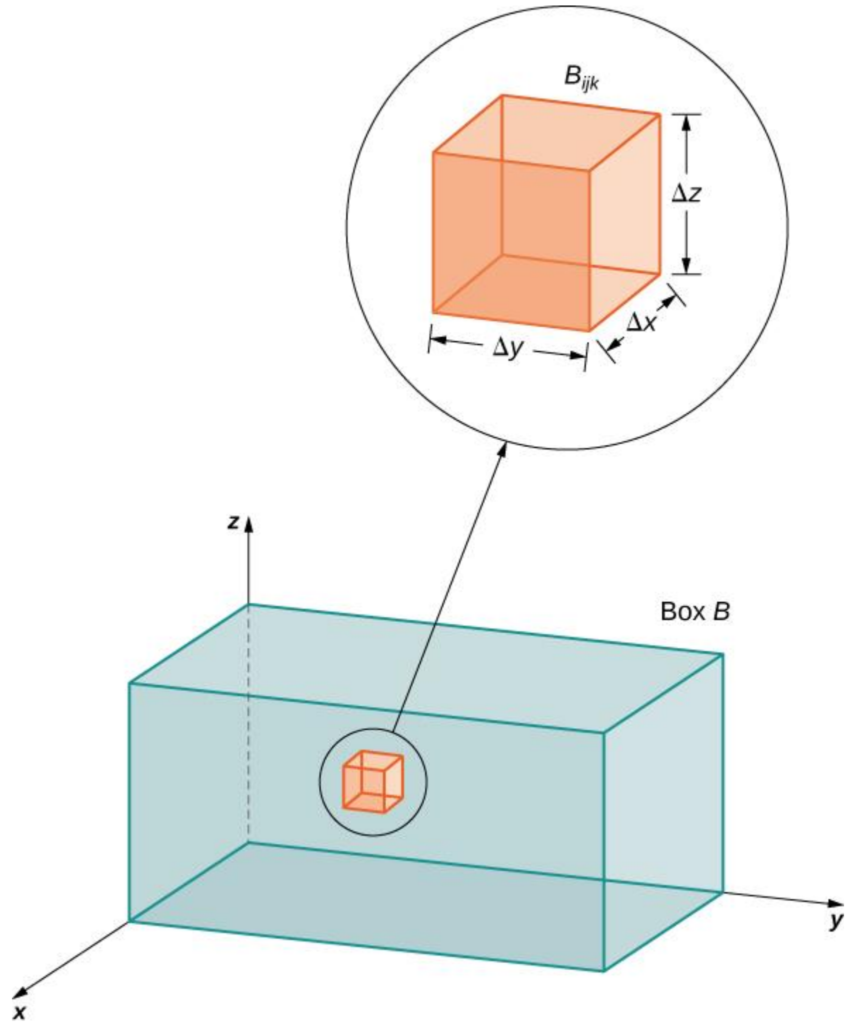
$$B = \{(x, y, z) \mid a \le x \le b, c \le y \le d, r \le z \le s\}.$$

We divide this box into sub-boxes of equal dimensions. We divide $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, $[c, d]$ into m subintervals $[y_{j-1}, y_j]$ of equal width Δy, and $[r, s]$ into n subintervals $[z_{k-1}, z_k]$ of equal width Δz. Thus we have sub-boxes given by

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

Each sub-box has volume $\Delta V = \Delta x \Delta y \Delta z$. Taking a sample point $({x_{ijk}}^*, {y_{ijk}}^*, {z_{ijk}}^*)$ in B_{ijk}, we may form the **triple Riemann sum**

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f({x_{ijk}}^*, {y_{ijk}}^*, {z_{ijk}}^*) \; \Delta V$$



We then define the triple integral as follows.

Definition: The **triple integral** of f over the box B is

$$\iiint_B f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f({x_{ijk}}^*, {y_{ijk}}^*, {z_{ijk}}^*)\ \Delta V$$

if the limit exists.

We also recover **Fubini's Theorem** for triple integrals.

If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$ then

$$\iiint_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz$$

Example: For $B = [2,3] \times [1,2] \times [0,1]$, evaluate the triple integral

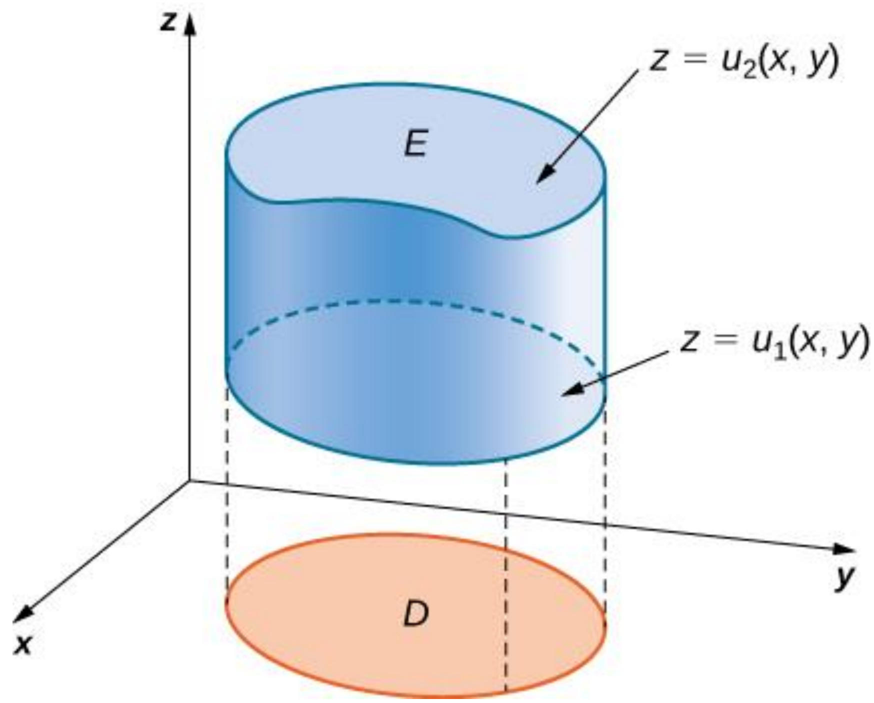
$$\iiint_B 8xyzdV$$

Using similar techniques as with the double integral, we may define the **triple integral over a general bounded region $\boldsymbol{E}$.** Recall in two variables we partition double integrals into two types (do you recall what those were?). For the triple integral, we have three types.

Type I: A solid region is said to be of type I if it lies between the graphs of two continuous functions of x and y, thus

$$E = \{(x, y, z) \mid (x, y) \epsilon D,\ u_1(x, y) \leq z \leq\ u_2(x, y)\}$$

where D is the projection of E onto the xy-plane as shown below.



In this case we have

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz\right] dA$$

This, in turn, bifurcates into the type I and type II plane regions we saw for the double integral. If the projection D is a type I planar region (so that D lies between the graphs of two continuous functions of x), where

$$E = \{(x, y, z) \mid a \leq x \leq b,\ g_1(x) \leq y \leq\ g_2(x),\ u_1(x, y) \leq z \leq\ u_2(x, y)\}$$

we may write

$$\iiint\limits_E f(x, y, z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz\, dydx$$

If the projection D is a type II planar region (so that D lies between the graphs of two continuous functions of y) where

$$E = \{(x, y, z) \mid c \leq y \leq d,\ h_1(y) \leq x \leq\ h_2(y),\ u_1(x, y) \leq z \leq\ u_2(x, y)\}$$

we may write

$$\iiint\limits_E f(x, y, z)dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)dz\, dxdy$$

Type II: A solid region is said to be of type II if it is of the form

$$E = \{(x, y, z) \mid (y, z) \epsilon D,\ u_1(y, z) \leq x \leq\ u_2(y, z)\}$$

where D is the projection of E onto the yz-plane. We then have

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)dx\right] dA$$

Type III: A solid region is said to be of type III if it is of the form

$$E = \{(x, y, z) \mid (x, z) \epsilon D,\ u_1(x, z) \leq y \leq\ u_2(x, z)\}$$

where D is the projection of E onto the yz-plane. We then have

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)dy\right] dA$$

For both type II and type III regions, there may be two possible expressions for the integral depending on whether D is a type I or type II plane region (as we saw in type I).

Example: Evaluate the triple integral of the function $f(x, y, z) = 5x - 3y$ over the solid tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Applications of the Triple Integral: Suppose that $f(x, y, z) = 1$ for all points in the region E. Then we have

$$V(E) = \iiint_E dV$$

Example: Determine the volume of the region that lies behind the plane $x + y + z = 8$ and in front of the region in the yz-plane that is bounded by $z = \frac{3}{2}\sqrt{y}$ and $z = \frac{3}{4}y$.

More Applications: All of the applications we did using double integrals can be applied in a similar manner to triple integrals. What would these look like? (electric charge, mass, moments, center of mass, moments of inertia)

Homework:

- Section 15.6: 1-21, 27-47 odds