15.4: Applications of Double Integrals.

OBJECTIVE

- Density and Mass/Electric Charge.
- Moment and Center of Mass.
- Moment of Inertia.

One application we have already seen is that double integrals allow us to compute volumes. We next turn to some applications in physics.

In single variable calculus, we were able to compute moments and center of mass of a thin plate or lamina with constant density. Now, equipped with double integrals, we can handle two dimensions and variable density.

Suppose we have a lamina occupying a region D in the xy-plane with density at a point (x, y) given by $\rho(x, y)$ where ρ is continuous on D. We find the total mass m of the lamina by

$$m = \int\int_D \rho(x, y) dA$$

Likewise, if an electric charge is distributed over a region D and the charge density is given by $\sigma(x, y)$ at a point (x, y) in D, then the total charge Q is given by

$$Q = \int\int_D \sigma(x, y) dA$$

Example: Electric charge is distributed over the disk $x^2 + y^2 \leq 1$ so that the charge density at (x, y) is $\sigma(x, y) = \sqrt{x^2 + y^2}$ (measured in Coulombs per square meter). Find the total charge on the disk.

Moments and Center of Mass: We may also use double integrals to find the center of mass of a lamina with *variable* density. Utilizing the definition of moments (from chapter 8) and applying our tool of double integration, we may obtain the moment of a lamina **about the x –axis:**

$$M_x = \int\int_D y\rho(x, y)dA$$

and **about the y –axis:**

$$M_y = \int\int_D x\rho(x, y)dA$$

Recall we define the center of mass by $(\bar{x}, \bar{y})$ so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$. What is the physical significance of this?

The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\int\int_D x\rho(x,y)dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\int\int_D y\rho(x,y)dA$$

where

$$m = \int\int_D \rho(x,y)dA$$

Example: Find the mass and center of mass of the lamina that occupies the region D bounded by $y = x + 2$ and $y = x^2$ with density function $\rho(x,y) = kx^2$.

Moment of Inertia: In physics, the moment of inertia of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. We can extend this concept to a lamina with density function $\rho(x, y)$ and occupying a region D by using the same techniques as for ordinary moments. We have the following results.

The **moment of inertia** of the lamina **about the x –axis:**

$$I_x = \int\int_D y^2 \rho(x, y) dA$$

The **moment of inertia** of the lamina **about the y –axis:**

$$I_y = \int\int_D x^2 \rho(x, y) dA$$

The **moment of inertia about the origin (or polar moment of inertia):**

$$I_0 = \int\int_D (x^2 + y^2) \rho(x, y) dA$$

Note that $I_x + I_y = I_0$.

Finally, we define the **radius of gyration of a lamina about an axis** as the number R such that $mR^2 = I$ where m is the mass of the lamina and I is the moment of inertia about the given axis. This says that if the mass of the lamina were concentrated at a distance R from the axis, then the moment of inertia of the "point mass" would be the same as for the lamina.

Example: Use the triangular region R given by the vertices $(0,0)$, $(2,2)$, and $(2,0)$, with density function $\rho(x, y) = xy$, find all three moments of inertia. Also find the radius of gyration w.r.t the x-axis, the y-axis, and the origin.

Homework:

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