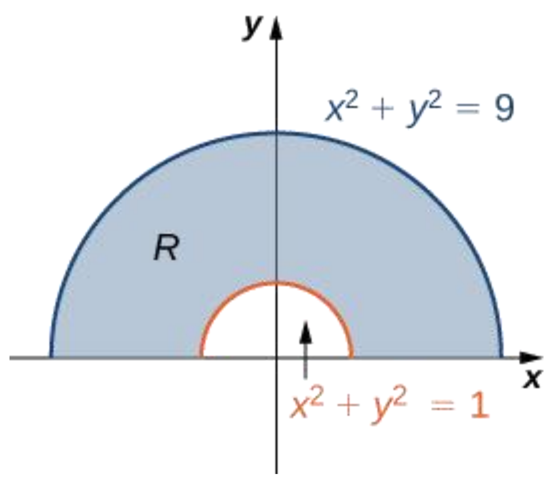
15.3: Double Integrals in Polar Coordinates.

OBJECTIVE

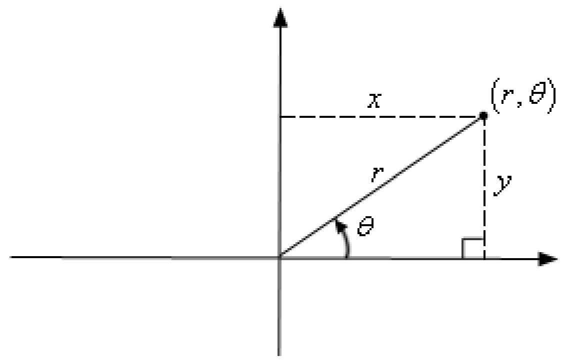
- Introduce Double Integrals in Polar Coordinates.

Imagine we are to evaluate a double integral over the region below



In Cartesian coordinates, what would the integral set up look like?

Polar Coordinates: Recall that in addition to Cartesian coordinates, we may denote points in the plane using polar coordinates. These are particularly useful when dealing with circular regions.



Recall the conversion formulas

$$x = r\cos\theta \qquad\qquad y = r\sin\theta \qquad\qquad x^2 + y^2 = r^2$$

To toggle back and forth between these coordinate systems, we use the following.

Changing Coordinates: If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b,\ \alpha \leq \theta \leq \beta$ where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\int\int_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta.$$

Caution! Don't forget the factor of r on the differential. Converting from Cartesian to polar requires replacing of dA with $rdrd\theta$.

Example: Evaluate the integral $\int\int_R \; 3x \, dA$ over the region $R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$.

Example: Determine the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$, and inside the cylinder $x^2 + y^2 = 5$.

General Polar Regions:

If f is continuous on a polar region of the form $D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$ then

$$\int\int_D f(x, y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta\,, r\sin\theta)rdrd\theta.$$

Example: Evaluate the integral $\int\int_D\, r^2 \sin\theta\, r\, drd\theta$ where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

Homework:

- Section 15.3: 1-31 odds