15.2: Double Integrals over General Regions.

OBJECTIVE

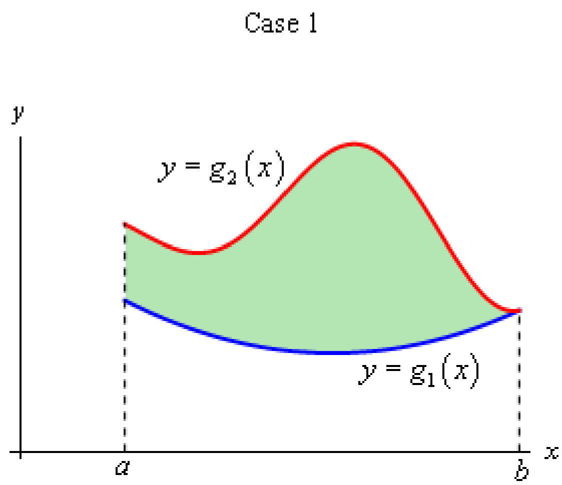
- Double Integrals over General Regions.

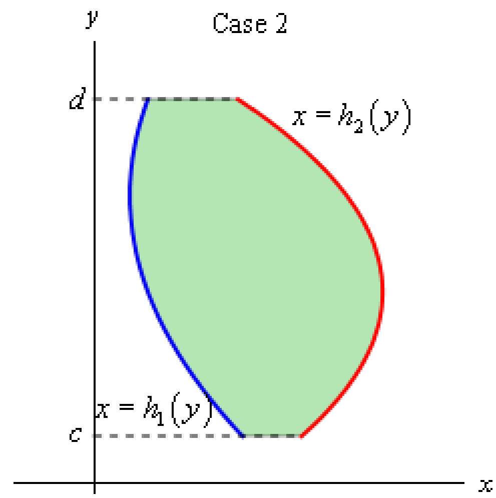
While it is useful to be able to compute double integrals over rectangular regions, most regions that concern us are not rectangular. We thus look at the double integral

$$\int\int_D f(x,y)dA$$

where D is any region. If we think about $f(x,y) \geq 0$ as the volume in the region D, how can we integrate the above expression over a rectangular region R?

We consider two cases.





In Case 1 the integral is defined to be

$$\int\int_D f(x,y)dA = \int_a^b \int_{g_2(x)}^{g_1(x)} f(x,y)dydx$$

where $D = \{(x,y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$.

In Case 2 the integral is defined to be

$$\int\int_D f(x,y)dA = \int_c^d \int_{h_2(y)}^{h_1(y)} f(x,y)dxdy$$

where $D = \{(x,y) \mid h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$.

Example: Evaluate the integral $\int\int_D (4xy - y^3)dA$ where D is the region bounded by $y = \sqrt{x}$ and $y = x^3$.

Note the importance of drawing the diagram!

While we may integrate our double integral in either order, it is often beneficial to choose one order of the other. Sometimes it won't even be possible to integrate using one order (but possible using the other!).

Example: Evaluate the integral $\int_0^3 \int_{x^2}^9 x^3 \; e^{y^3} \, dydx$.

Example: Find the volume of the solid that lies below the surface $z = 16xy + 200$ and lies above region in the xy-plane bounded by $y = x^2$ and $y = 8 - x^2$.

Properties of Double Integrals: Assuming all of the following integrals exist, we have the following properties.

1. $\int\int_D [f(x,y) + g(x,y)]dA = \int\int_D f(x,y)dA + \int\int_D g(x,y)dA$

2. $\int\int_D cf(x,y)dA = c\int\int_D f(x,y)dA$ where c is a constant.

3. If $f(x,y) \geq g(x,y)$ for all (x,y) in D then

$$\int\int_D f(x,y)dA \geq \int\int_D g(x,y)dA.$$

4. If $D = D_1 \cup D_2$, where the interiors of D_1 and D_2 do not intersect, then

$$\int\int_D f(x,y)dA = \int\int_{D_1} f(x,y)dA + \int\int_{D_2} f(x,y)dA.$$

Finally, if we integrate the constant function 1 over a region D, we get the area of D.

$$\int\int_D 1dA = A(D).$$

Homework:

- Section 15.2: 1-31, 35-39, 45-57 odds