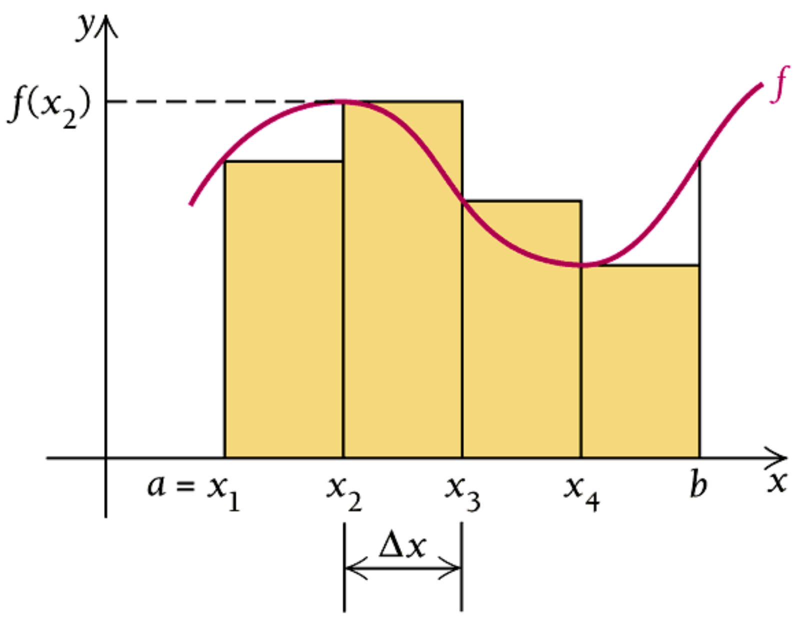
15.1: Double Integrals over Rectangles.

OBJECTIVE

- Compute Double Integrals over Rectangles.
- Introduce the Midpoint Rule.
- Iterated Integrals and Fubini's Theorem.
- Find the Average Value.

Recall from calculus that the definite integral of a function of a single variable over the interval $[a, b]$.

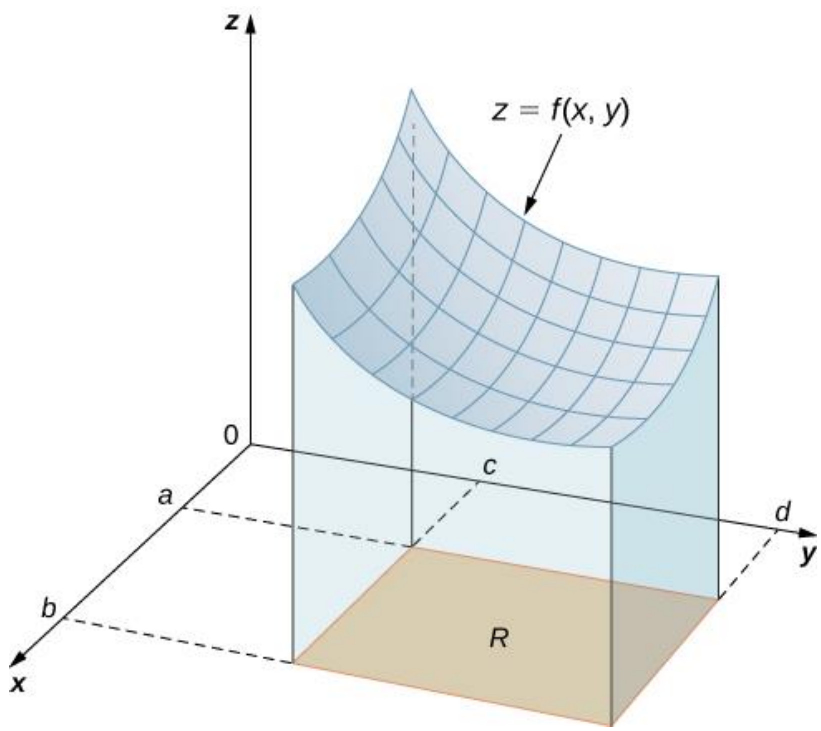


Volumes and Double Integrals:

Similar to the above discussion, we may consider a function of two variables defined on the closed rectangle given by

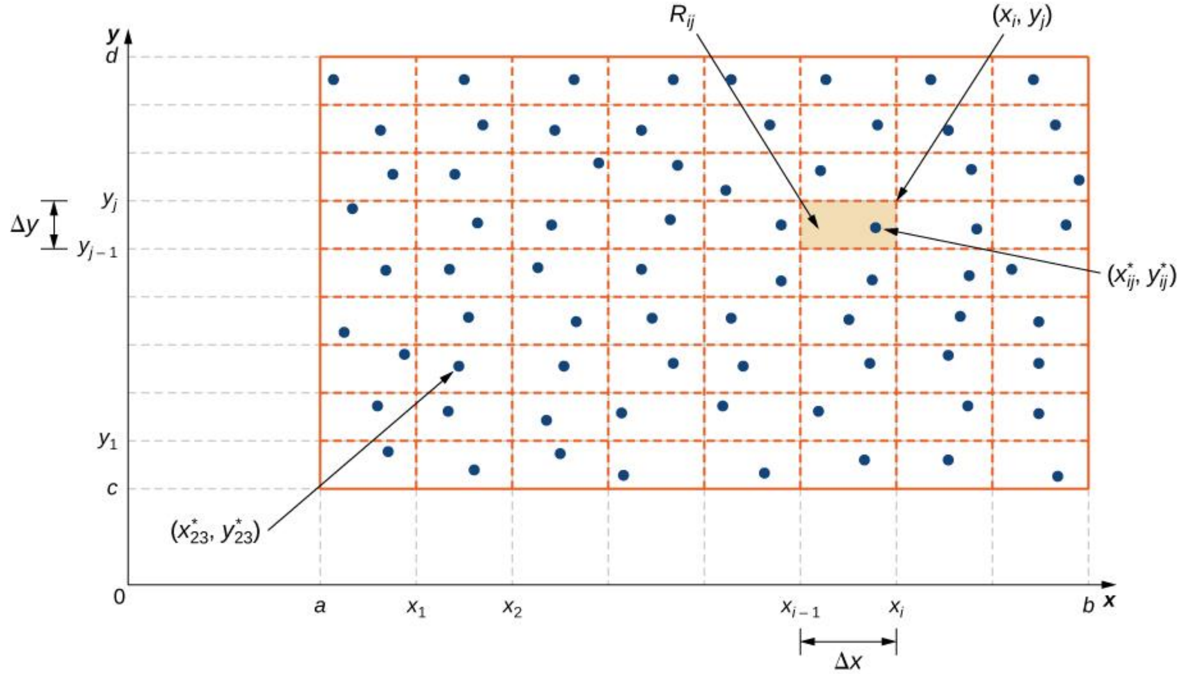
$$R = [a, b] \times [c, d] = \{a \leq x \leq b, c \leq y \leq d\}$$

where, for simplicity, we will momentarily assume if $f(x, y) \geq 0$.



If we let S denote the region above R and below the graph of f above R, then our goal is to find the volume of this region.

We achieve this by dividing R up into subrectangles. In particular, we divide $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{m}$ and $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = \frac{d-c}{n}$.



This allows us to form the subrectangles given by

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{ (x, y) \mid x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\}$$

each which has area $\Delta A = \Delta x \Delta y$.

If we choose a sample point $(x_{ij}{}^*, y_{ij}{}^*)$ in each R_{ij}, we can approximate the volume of our "column" by the each

$$base \ \times height = f(x_{ij}{}^*, y_{ij}{}^*)\Delta A.$$

Thus, we may approximate our total volume S by

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}{}^*, y_{ij}{}^*)\Delta A$$

Definition: The **double integral** of f over the rectangle R is

$$\iint f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}{}^{*}, y_{ij}{}^{*}\right) dA$$

if the limit exists.

If the above limit exists, we say f is **integrable**. All continuous functions are integrable. The sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}{}^{*}, y_{ij}{}^{*}\right) \Delta A$$

is called the **double Riemann sum**.

Fact: If $f(x,y) \geq 0$ then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint f(x,y)\,dA.$$

The Midpoint Rule:

Just as in one variable, there are approximating methods for finding double integrals. Here we will discuss one of them, the midpoint rule. We use the above rectangular columns where our sample point $\left(x_{ij}{}^{*}, y_{ij}{}^{*}\right)$ in R_{ij} is chosen to be the center $(\bar{x}_i, \bar{y}_i)$.

Example: Use the midpoint rule with$m = n = 2$ to estimate the value of the integral $\iint x - 3y^2\, dA$ over the rectangular region

$$R = \{\ (x, y)\ |\ 0 \leq x \leq 2, 1 \leq y \leq 2\}.$$

Iterated Integrals: Just as we use the F.T.O.C. for one variable to compute integrals, we seek simpler methods of integration beyond the definition. We achieve this by breaking the double integral into single integrals and integrating first with respect to one variables and then with respect to the other.

Definition: The **iterated integral** for a function $f(x, y)$ over the rectangular region $[a, b] \times [c, d]$ is given by

$$\int_a^b \int_c^d f(x, y) dydx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx \text{ or}$$

$$\int_c^d \int_a^b f(x, y) dxdy = \int_c^d \left[\int_a^b f(x, y) dx \right] dy$$

This tells us that we work from "the inside out".

Example: Evaluate the integral $\int_1^3 \int_1^5 \frac{\ln y}{xy} dydx$ as an iterated integral.

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Did the order of the iterated integral matter?

The Average Value: In a similar manner as in one variable, one can show that the average value of a function f of two variables defined on a rectangle R is given by

$$f_{ave} = \frac{1}{A(R)} \iint f(x,y)dA$$

Homework:

- Section 15.1: 1-3, 9-43, 47 odds