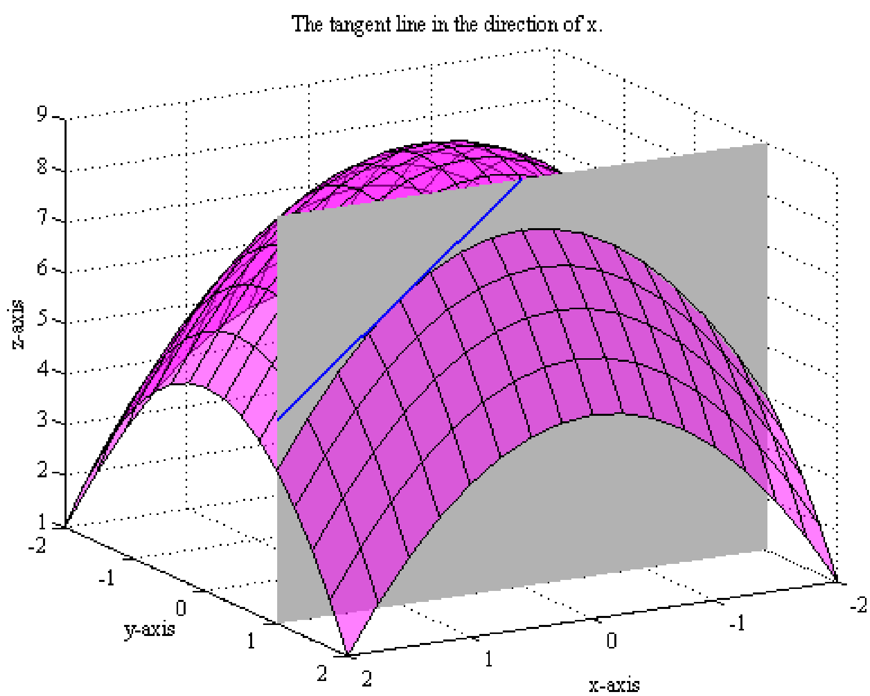
14.6: Directional Derivatives and the Gradient Vector.

OBJECTIVE

- Introduce the directional derivative and maximize.
- Introduce the gradient vector and how to find tangent planes to level surfaces.

Recall that we previously defined f_x and f_y at a point (x_0, y_0) to be given by:

What direction do these partial derivative represent?



Our goal is to generalize the definitions of f_x and f_y to f_u where $\boldsymbol{u} = \langle a, b \rangle$ is an arbitrary unit vector.

Definition: The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\boldsymbol{u} = \langle a, b \rangle$ is

$$D_{\boldsymbol{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

This definition is great, but in practice we would like something more practical. One can show the above definition gives the following result.

Theorem: If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\boldsymbol{u} = \langle a, b \rangle$ and

$$D_{\boldsymbol{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

In particular, if our unit vector $\boldsymbol{u}$ makes an angle θ with the positive x-axis, we may write $\boldsymbol{u} = \langle \cos\,\theta\,, \sin\,\theta \rangle$.

Example: Find $D_{\boldsymbol{u}}f(2,0)$ where $f(x,y) = xe^{xy} + y$ and $\boldsymbol{u}$ is the unit vector in the direction of $\theta = \frac{2\pi}{3}$.

Example: Find $D_{\boldsymbol{u}}f(x,y,z)$ where $f(x,y,z) = x^2z + y^3z^2 - xyz$ in the direction of $\boldsymbol{u} = \langle -1,0,3 \rangle$.

The Gradient Vector: Notice the following relation. We may write

$$D_{\boldsymbol{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b = \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a,b\rangle$$

where $= \langle a,b\rangle$. This is one motivation for the following definition.

> **Definition:** If f is a function of two variables x and y, then the **gradien**t of f is the vector function ∇f defined by
>
> $$\nabla f\ (x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

With this definition we may write

$$D_{\boldsymbol{u}}f(x,y) = \nabla f\ (x,y) \cdot \boldsymbol{u}$$

How do you think this generalizes to three variables?

Example: Find $D_{\boldsymbol{u}} f(x, y, z)$ where $f(x, y, z) = \sin yz + \ln x^2$ at $(1,1,\pi)$ in the direction of $\boldsymbol{u} = \langle 1,1,-1 \rangle$.

Maximizing the directional derivative (the path of steepest descent):

> **Theorem:** Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\boldsymbol{u}} f(\boldsymbol{x})$ is $|\nabla f(\boldsymbol{x})|$ which occurs when $\boldsymbol{u}$ has the same direction as the gradient vector $\nabla f(\boldsymbol{x})$.

Example: Find the maximum rate of change of $f(x, y, z) = x \ln yz$ at the point $\left(1,2,\frac{1}{2}\right)$ and the direction which it occurs.

Fact: The gradient vector $\nabla f(\boldsymbol{x})$ is orthogonal to the level curve $f(\boldsymbol{x}) = k$ at the point $\boldsymbol{x}_0$.

Example: Find equations of the tangent plane and normal line to the given surface at the point $(1,2,1)$.

$$xy + yz + zx = 5$$

Homework:

- Section 14.6: 5-33, 37-57 odds