14.5: The Chain Rule in Several Variables.

OBJECTIVE

- Introduce the chain rule in several variables.
- Describe implicit differentiation using the chain rule.

Recall that the chain rule in one variable is given by

$$\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big)g'(x).$$

In two variables, we obtain different versions depending on the nature of the composite functions.

If $z = f(x, y)$ where x and y are functions of a variable t, that is,

$$z = f(x, y) = f(x(t), y(t))$$

then, assuming f is differentiable, which is the case when f_x and f_y are continuous, then we have the following rule.

The Chain Rule (Case 1): Suppose $z = f(x, y)$ is a differentiable function of x and y with $x = g(t)$ and $y = h(t)$, both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example: Find $\frac{dz}{dt}$ if $z = f(x, y) = x^2 - 3xy + 2y^2$ if $x(t) = 3\cos 2t$ and $(t) = 4\sin 2t$.

If $z = f(x, y)$ where x and y are functions of two variables s and t so that

$$z = f(x, y) = f(x(s, t), y(s, t))$$

then, assuming f is differentiable and x and y are differentiable functions of s and t, we have the following rule.

The Chain Rule (Case 2): Suppose $z = f(x, y)$ is a differentiable function of x and y with $x = g(s, t)$ and $y = h(s, t)$, both differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example: Use the chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ if $z = \sqrt{x}e^{xy}$ and $x = 1 + st$ and $y = s^2 - t^2$.

How do you think the chain rule generalizes to a dependent variable u as a function of n intermediate variables each of which is a function of m independent variables?

Example: Use the chain rule to find $\frac{\partial T}{\partial p}$, $\frac{\partial T}{\partial q}$, $\frac{\partial T}{\partial r}$ if $T = \frac{v}{2u+v}$ and $u = pq\sqrt{r}$, $v = pr\sqrt{q}$ when $p = 2, q = 1$, and $r = 4$.

Implicit Differentiation Revisited: We can utilize the chain rule to give a more complete description of implicit differentiation. In particular, we write $F(x, y) = 0$ and suppose that this implicitly defines y as a differentiable function of x so that we have $F(x, y) = F(x, f(x)) = 0$. Supposing F is differentiable allows us to use the chain rule in case 1. Then, differentiating both sides w.r.t. x gives us

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

where solving for $\frac{dy}{dx}$ gives us

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

The conditions for this assumption come from the **Implicit Function Theorem** which is beyond the scope of this class but are sufficient for the cases we will have.

Example: Use the above equation to find $\frac{dy}{dx}$ if $\sin xy = 1 + \cos y$.

If z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$ so that $F(x, y, z) = F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f where F and f are differentiable functions, then in a similar manner as above we can use the chain rule to derive the formulas

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Example: Use the above equation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$x^2 + y^2 - 2xyz = 4.$

Homework:

- Section 14.5: 1-13, 21-35 odds