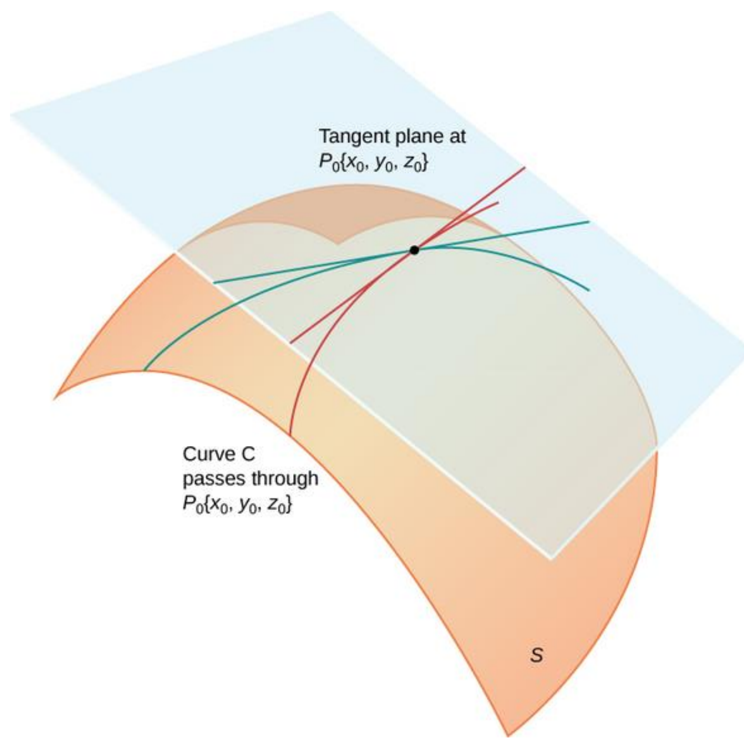
14.4: Tangent Planes and Linear Approximations

OBJECTIVE

- Introduce Tangent Lines and Linear Approximations.

Recall that in one variable, if we zoom in toward a point on the graph of a differentiable function, the graph becomes indistinguishable from its tangent line. Additionally, we can approximate the function at a given point by a linear equation (the tangent line).

In two variables, we have a surface instead of a line. If we zoom in toward a point on the graph of a differentiable function in two variables, the graph becomes indistinguishable from its **tangent** plane. We can approximate this with a linear function of two variables.



Recall the equation for a plane in 3D.

For a tangent plane to exist at a point, it is sufficient for the function that defines the surface to be differentiable at the point.

Suppose f has continuous partial derivatives. A equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example: Find the equation of the tangent plane to the surface defined by the function

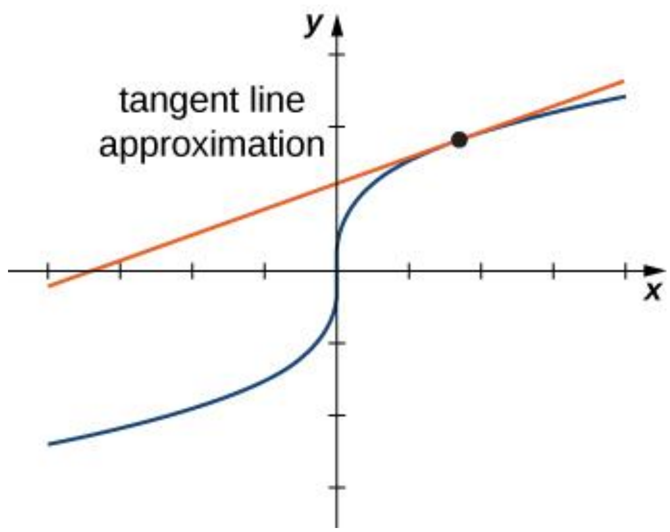
$$f(x, y) = x^3 - x^2y + y^2 - 2x + 3y - 2$$

at the point $(-1,3)$.

Linear Approximations:

Recall that for a function $y = f(x)$ of one variable, we can make a linear approximation of f at a point a using the equation

$$y \approx f(a) + f'(a)(x - a).$$



In two variables, we replace the tangent line with a tangent plane, but the same idea of approximation holds.

Linear Approximation: Given a function $z = f(x, y)$ with continuous partial derivatives at the point (a, b), the linearization of f at the point (a, b) is given by

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

and

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linear approximation of f at (a, b).

Example: Given the function $f(x, y) = e^{5-2x+3y}$, approximate $f(4.1, .9)$ using the point $(4,1)$.

Differentiability: We saw that a for a tangent plane to exist at a point, it must have partial derivatives at that point. This, however, is not sufficient to guarantee "smoothness" at the point. For this we need the idea of differentiability in two variables.

Definition: If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Essentially, this says that a differentiable function is one in which the linear approximation is a good approximation when (x, y) is near (a, b).

Here's an easier way of explicitly checking differentiability for many cases.

Theorem: If the partial derivatives f_x and f_y exist near a point (a, b) and are continuous at (a, b) then f is differentiable at (a, b).

Example: Explain why the function $f(x, y) = \frac{1+y}{1+x}$ is differentiable at the point $(1,3)$. Then find the linearization approximate $L(x, y)$ of the function at this point.

Differentials: Recall that in one variable, for a differentiable function $y = f(x)$, we define the differential by $dy = f'(x)dx$. We can do the same in two variables.

Definition: The **differential** dz is defined by

$$dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Example: Find the differential of the function $L = xze^{-y^2-z^2}$.

Homework:

- Section 14.4: 1-5, 11-15, 25-35 odds