14.3: Partial Derivatives.

OBJECTIVE

- Introduce Partial Derivatives.

If we are given a function $f(x,y)$ of two variables, we can fix one variable, say $y = b$, and let x vary. In this case we have a function of *one variable* $g(x) = f(x,b)$. If g has a derivative at a, then we say it has **partial derivative of $\boldsymbol{f}$ with respect to $\boldsymbol{x}$ at $(\boldsymbol{a}, \boldsymbol{b})$** and denote it by $f_x(a,b) = g'(a)$ where $g(x) = f(x,b)$. When we take a partial derivative with respect to one variable, we treat the other variable as a constant.

The definition of the partial derivatives for $f(x,y)$ are as follows:

Notation: Let $z = f(x, y)$ then we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Sometimes we call these the **first partial derivatives**.

Rules for Finding the Partial Derivatives of $\boldsymbol{z = f(x, y)}$:

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ w.r.t. x.

2. To find f_y, regard x as a constant and differentiate $f(x, y)$ w.r.t. y.

Example: Find the first partial derivatives of the function

$$f(x, y) = \frac{x}{(x + y)^2}$$

Functions of more than two variables:

For functions of more variables, we apply the same reasoning to find partial derivatives of such functions.

Example: Find all first order partial derivatives of the function

$$F(x, y, z) = x^2 e^{-yz} + y^3 e^x$$

Implicit differentiation:

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for

$$x^3 z^2 - 5xy^5 z = x^2 + y^3$$

Higher Derivatives:

If f is a function of two variables, then the partial derivatives are also a function of two variables. Taking the partial derivatives again, we obtain the **second partial derivatives**. If $z = f(x, y)$, what are the possible second partial derivatives?

Example: Find all second order partial derivatives of the function

$$f(x, y) = \sin(3x) + x^2 e^{2y} + y^3$$

How are f_{xy} and f_{yx} related?

Clairaut's Theorem: Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

We can extrapolate this idea to any order partial derivative.

Example: Find $\frac{\partial^3 V}{\partial r \partial s \partial t}$ partial derivative.

$$V = \ln(r + s^2 + t^3)$$

Homework:

- Section 14.3: 5-9, 15-71 odds