13.3: Arc Length and Curvature:

OBJECTIVE

- To find the length of a given space curve.
- To parametrize curves, describe curvature and define normal and binormal vectors.

To begin, let's remember the definition of the length of a **plane curve** with parametric equations $x = f(t)$ and $y = g(t)$ (supposing both are continuous):

By parametrizing $x = f(t)$, $y = g(t)$ and $z = h(t)$, presuming continuous derivatives and one transversal as t increases from a to b, we have

$$L = \int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2 + \left(h'(t)\right)^2}\, dt$$

Example: Find the length of the curve

$$\boldsymbol{r}(t) = 2t\mathbf{i} + 3\sin 2t\,\mathbf{j} + 3\cos 2t\,\mathbf{k}. \qquad 0 \le t \le \sqrt{10}$$

Setting $b = t$ in the above integral, we may view $s(t) = L(t) = L$ as a function of t.

Example: For the above example, find the arc length function $s(t)$.

So what? Solving the above function for t, we may **reparametrize** our function in the form

$$\boldsymbol{r}(t(s)) =$$

(It turns out there are many different ways to parametrize a curve. One can show that regardless of the parametrization, the arc length will be the same.)

What's cool about this is that we now have a parametrization that tells us, after traversing s units on the curve, where we are!

Example: For the above space curve, where are we on the curve after we have traveled $\frac{\pi\sqrt{10}}{3}$ units?

Curvature: Recall that a curve given by $\boldsymbol{r}(t)$ is smooth if $\boldsymbol{r}'(t)$ is continuous and not equal to 0. The curvature will then measure how quickly a curve is changing direction at a given point.

With this in mind, we define the curvature to be the magnitude of the rate of change of the unit tangent vector w.r.t. the arc length.

What's so nice about defining it w.r.t. the arc length?

The **curvature** of a curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$

where $\mathbf{T}$ is the unit tangent vector. As a function of t we have

$$\kappa(t) = \frac{|\mathbf{T}`(\mathrm{t})|}{|\boldsymbol{r}`(t)|}.$$

Or, if you really want to keep it simple and stick with $\boldsymbol{r}(t)$ only, we have

$$\kappa(t) = \frac{|\boldsymbol{r}`(t)\times\boldsymbol{r}``(t)|}{|\boldsymbol{r}`(t)|^3}.$$

Example: Find the curvature of $\boldsymbol{r}(t) = t\mathbf{i} + 3\sin t\,\mathbf{j} + 3\cos t\,\mathbf{k}$**.**

If we have a plane curve that we can write as $y = f(x)$, we can use the above equations to derive the simpler formula

$$\kappa(x) = \frac{|f''(x)|}{|1 + (f'(x))^2|^3}.$$

Example: Find the curvature of $y = x^4$ at the points (0,0), (1,1) and (2,16).

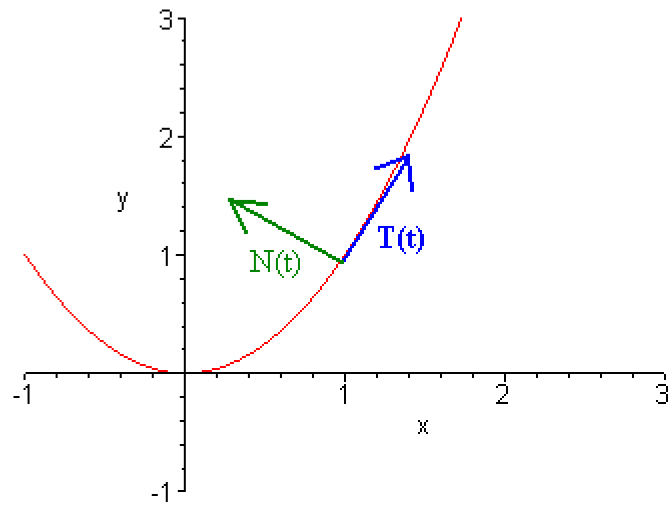
Normal and Binormal Vectors:

The **unit normal vector** is defined by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.$$

Where on earth does that come from? Well, this vector is orthogonal (or normal) to the unit tangent vector and thus the curve as well and we recall from our discussions about the equations of planes that normal vectors can be very informative.

This comes from the fact that if $|\boldsymbol{r}(t)|$ is a constant for all t, then $\boldsymbol{r}'(t)$ is orthogonal to $\boldsymbol{r}(t)$. (Show this!)



The **unit binormal vector** is defined by

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

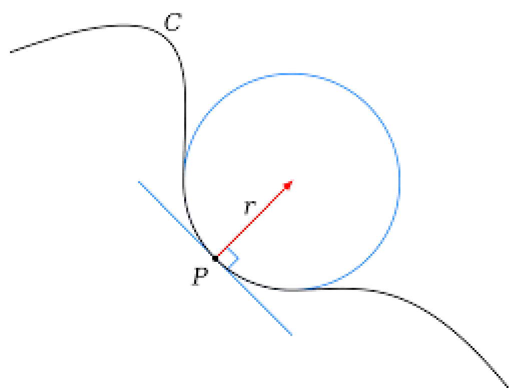
By the way, what can we say about the cross product of two vectors?

Example: Find the normal and binormal vectors for

$$\boldsymbol{r}(t) = t\mathbf{i} + 3\sin t\,\mathbf{j} + 3\cos t\,\mathbf{k}.$$

The **normal plane** of a point *P* on a curve *C* is the plane determined by the normal and binormal vectors $\mathbf{N}$ and $\mathbf{B}$ at *P.* The **osculating plane** of *C* at *P* is the plane determined by the vectors $\mathbf{T}$ and $\mathbf{N}$.

The **osculating circle** is the circle that lies in the osculating plane of *C* at *P* that has the same tangent as *C* at *P* and lies on the concave side of *C* with radius $\rho = \frac{1}{\kappa}$.



Example: Find and graph the osculating circle of $y = x^2$ at the origin.

Homework:

- Section 13.3: 1-29 odds