13.2: The Calculus of Vector Functions:

OBJECTIVE

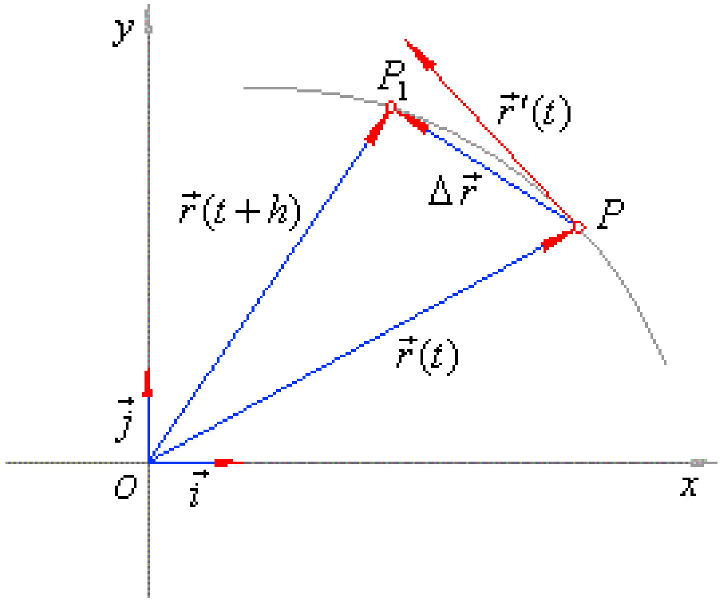
- Introduce the Derivative of a Vector Function and its Differentiation Rules.
- Introduce the Integral of a Vector Function.

To begin, let's remember the definition of the derivative of a real-valued function:

Derivative of a Vector-Valued Function:

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{r}'(t) = \lim_{h\to 0}\frac{\boldsymbol{r}(t+h)-\boldsymbol{r}(t)}{h}$$

provided the limit exists. We call $\boldsymbol{r}'(t)$ is the tangent vector of the curve.



It is often useful to also consider the **unit tangent vector** given by

$$\mathbf{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}.$$

> If $\boldsymbol{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ where f, g and h are differentiable functions, then
>
> $$\boldsymbol{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Example: Find the derivative and unit tangent vector $\mathbf{T}(t)$ of

$$\boldsymbol{r}(t) = tan^{-1}\,t\,\mathbf{i} + 2e^{2t}\mathbf{j} + 8te^{t}\mathbf{k}.$$

Example: For the above example, find $\boldsymbol{r}''(t)$.

Example: Sketch the plane curve below with the given vector equation. Find $\boldsymbol{r}'(t)$ and sketch the position vector $\boldsymbol{r}(t)$ and the tangent vector $\boldsymbol{r}'(t)$ for the given value of t.

$$\boldsymbol{r}(t) = (\cos t + 1)\mathbf{i} + (\sin t - 1)\mathbf{j} \quad \text{at} \quad t = -\frac{\pi}{3}.$$

Differentiation Rules:

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be differentiable vector functions, c a scalar and f a real-valued function. Then

1. $\frac{d}{dt}[\boldsymbol{u}(t) + \boldsymbol{v}(t)] = \boldsymbol{u}'(t) + \boldsymbol{v}'(t)$

2. $\frac{d}{dt}[c\boldsymbol{u}(t)] = c\boldsymbol{u}'(t)$

3. $\frac{d}{dt}[f(t)\boldsymbol{u}(t)] = f'(t)\boldsymbol{u}(t) + f(t)\boldsymbol{u}'(t)$

4. $\frac{d}{dt}[\boldsymbol{u}(t) \cdot \boldsymbol{v}(t)] = \boldsymbol{u}'(t) \cdot \boldsymbol{v}(t) + \boldsymbol{u}(t) \cdot \boldsymbol{v}'(t)$

5. $\frac{d}{dt}[\boldsymbol{u}(t) \times \boldsymbol{v}(t)] = \boldsymbol{u}'(t) \times \boldsymbol{v}(t) + \boldsymbol{u}(t) \times \boldsymbol{v}'(t)$

6. $\frac{d}{dt}[\boldsymbol{u}(f(t))] = f'(t)\boldsymbol{u}'(f(t))$

Example: Find the parametric equations for the tangent line to the curve given by the parametric equations $x = \ln(t+1)$, $y = t\cos 2t$, $z = 2^t$ at the point (0,0,1).

The Integral of a Continuous Vector-Valued Function:

$$\int_a^b \boldsymbol{r}(t)\,dt = \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k}.$$

For continuous vector functions, the Fundamental Theorem of Calculus says

$$\int_a^b \boldsymbol{r}(t)\,dt = \mathbf{R}(\mathrm{b}) - \mathbf{R}(\mathrm{a})$$

where $\mathbf{R}$ is the antiderivative of $\boldsymbol{r}(t)$.

Example: Evaluate the definite integral.

$$\int_0^{\frac{\pi}{4}} (\sec t \tan t\, \mathbf{i} + \sin 2t \cos 2t)\mathrm{dt}$$

Homework:

- Section 13.2: 1-27 & 35-41 odds